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Research Article

An Improved Iterative Scheme using Successive Over-relaxation for Solution of Linear System of Equations

Zaheer Ahmed, Zubair Ahmed Kalhoro*, Abdul Wasim Shaikh, Muhammad Shakeel Rind Baloch, and Owais Ali Rajput

Institute of Mathematics and Computer Science, University of Sindh, Jamshoro, Pakistan

Abstract: To solve the system of linear equations is one of the hottest topics in iterative methods. The system of linear equations occurs in business, engineering, social and in sensitive research areas like medicine, therefore applying efficient matrix solvers to such systems is crucial. In this paper, an improved iterative scheme using successive overrelaxation has been constructed. The proposed iterative method converges well when a linear system's matrix is M-matrix, Symmetric positive definite with some conditions, irreducibly diagonally dominant, strictly diagonally dominant, and H-matrix. Such type of linear system of equations does arise usually from ordinary differential equations and partial differential equations. The improved iterative scheme has decreased spectral radius, improved stability and reduced the number of iterations. To show the effectiveness of the improved scheme, it is compared with the refinement of generalized successive over-relaxation and generalized successive over-relaxation method with the help of numerical experiments using MATLAB software.

Keywords: GSOR, RGSOR, Diagonally Dominant, Irreducibly Diagonally Dominant, Rate of Convergence

1. INTRODUCTION

The world of mathematics is surrounded by many mathematical problems. The efforts to find the solution to the system of linear equations are one of the most popular and interesting problems in the math world. A lot of people, apart from mathematicians, like computer scientists, chemists, biologists, physicists, engineers, social scientists, industry experts, economists etc, struggle to solve the system of linear equations in their fields [1-2]. The branch of mathematics which is devoted to developing a different algorithm for solving a system of linear equations is called Linear Algebra (LA) and the branch of LA that deals with the numerical solution of these linear systems, is called Numerical Linear Algebra (NLA). A linear equation system can be transformed into a matrix equation in the following form:

$$Ay=b \tag{1}$$

In equation 1, matrix A is an invertible square matrix whereas y and b are unknown and known column vectors respectively.

In NLA we search for the indirect or numerical solution of the linear system. The indirect methods are applied when the co-efficient matrices of matrix equations are large dimensional and sparse [3].

Improvements in the classical iterative techniques have been done earlier [11], however since the SOR outperforms all the other iterative techniques, an improvement in the SOR scheme will be highly effective. In this research work an iterative scheme namely "improved iterative scheme using successive overrelaxation (IIS)" is developed. This scheme is just an improved version of the generalized successive overrelaxation (RGSOR) refinement. Stability and Spectral Radius are used as comparative factors for checking the efficiency of the proposed algorithm. The IIS has decreased spectral radius, improved stability and

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*Corresponding author: Zubair Ahmed Kalhoro <zubair.kalhoro@usindh.edu.pk>

the number of iterations. The convergence of IIS has been proved. Different types of numerical experiments are considered to demonstrate the efficiency of the IIS method.

2. MATERIALS AND METHODS

First, we decompose the matrix $\boldsymbol{A} = \boldsymbol{T_m} - \boldsymbol{L_m} - \boldsymbol{U_m}$. Let $\boldsymbol{A} = (\boldsymbol{\alpha_{ij}})$ be a nonsingular square matrix and $\boldsymbol{T_m} = \boldsymbol{t_{ij}}$ with bandwidth 2m+1 is be a banded matrix, defined as

$$\boldsymbol{t_{ij}} = \begin{cases} \boldsymbol{\alpha_{ij}}, |\boldsymbol{i}-\boldsymbol{j}| \le \boldsymbol{m} \\ \boldsymbol{0} \quad \textbf{otherwise} \end{cases} \qquad (1.1)$$

Where $\boldsymbol{L_m}$ and $\boldsymbol{U_m}$ are strictly lower part and strictly upper part, these matrices are defined as under

$$\boldsymbol{T_m} = \begin{bmatrix} \alpha_{1,1} & \cdots & \alpha_{1,m+1} & \\ \vdots & \ddots & \ddots & \alpha_{n-m,n} \\ \alpha_{m+1,1} & \ddots & \ddots & \vdots \\ & \alpha_{n,n-m} & \cdots & \alpha_{n,n} \end{bmatrix}$$

$$\boldsymbol{L_m} = \begin{bmatrix} -\alpha_{m+2,1} & & & \\ \vdots & \ddots & & \\ -\alpha_{n,1} & \cdots & -\alpha_{n-m-1,n} & \end{bmatrix}$$

$$\boldsymbol{U_m} = \begin{bmatrix} & -\alpha_{1,m+2} & \cdots & -\alpha_{1,n} \\ & & \ddots & \vdots \\ & & & -\alpha_{n-m-1,n} \\ & & & \end{bmatrix}$$

Definition 1: [4-5] A square matrix $\boldsymbol{A} = (\boldsymbol{\alpha_{ij}})$ is known as diagonally dominant (DD) if

$$|\boldsymbol{\alpha_{ii}}| \ge \sum_{\substack{j=1 \\ j\neq 1}}^{n} |\boldsymbol{\alpha_{ij}}| \qquad (1.2)$$

Definition 2: [4-5] A square matrix $\boldsymbol{A} = (\boldsymbol{\alpha_{ij}})$ is known as strictly diagonally dominant (SDD) if

$$|\boldsymbol{\alpha_{ii}}| > \sum_{\substack{j=1 \\ j\neq 1}}^{n} |\boldsymbol{\alpha_{ij}}| \qquad (1.3)$$

Definition 3: [4-5] With satisfying the following four axioms a matrix $\boldsymbol{A}$ is called an $\boldsymbol{M}$-matrix.

i. $\boldsymbol{\alpha_{ii}} > \boldsymbol{0}$ for $\boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2}, \boldsymbol{3}, \ldots, \boldsymbol{n}$

ii. $\boldsymbol{\alpha_{ij}} \le \boldsymbol{0}$ for $\boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2}, \boldsymbol{3}, \ldots, \boldsymbol{n}$

iii. $\boldsymbol{A}$ must be a nonsingular

iv. $\boldsymbol{A^{-1}} > \boldsymbol{0}$

Definition 4: [4-5] If a square matrix $\boldsymbol{A}$ satisfies the following conditions then it is called symmetric positive definite.

i. $\boldsymbol{A^t} = \boldsymbol{A}$

ii. $\boldsymbol{y^t A y} > \boldsymbol{0}$

Definition.5: [4-5] Let $\boldsymbol{A}$ a square matrix and λ be its Eigen value then the equation $\boldsymbol{\rho(A)} = \mathbf{max}|\boldsymbol{\lambda}|$ is called spectral radius $\boldsymbol{A}$.

3. GENERALIZED SOR METHOD

The generalized SOR method for solving the system of linear equations is presented by Manideep Saha and Jahnvi Chakrabarty [6]. Using this method eq. (1) can be written as:

$$y^{(k+1)} = (T_m - \omega L_m)^{-1}\big((1-\omega)T_m + \omega U_m\big)y^{(k)} + (T_m - \omega E_m)^{-1}\omega b \qquad (2)$$

4. REFINEMENT OF GENERALIZED SOR METHOD

Hailu Muleta and Genanew Gofe proposed a refinement of the GSOR method [1]. Multiplying eq. (1) with $\boldsymbol{\omega}$ then substituting $\boldsymbol{A}$ with its splitting

$$\boldsymbol{\omega(T_m - L_m - U_m) = \omega y b} \qquad (2.1)$$

After simplification:

$$\boldsymbol{y = y + (T_m - \omega L_m)^{-1}(b - Ay)\omega} \qquad (2.2)$$

That is:

$$\boldsymbol{\tilde{y}^{(k+1)} = y^{(k+1)} + (T_m - \omega L_m)^{-1}(b - Ay^{(k+1)})\omega} \qquad (3)$$

Putting the values of $\mathbf{y^{(k+1)}}$ from Eq. (2) in Eq. (3) and after solving the refinement of the GSOR Scheme will be:

$$\boldsymbol{y^{(k+1)} = [(T_m - \omega L_m)^{-1}((1-\omega)T_m + \omega U_m)]^2 y^{(k)} + (T_m - \omega L_m)^{-1}[I + (T_m - \omega L_m)^{-1}((1-\omega)T_m + \omega U_m]\omega b} \qquad (4)$$

5. PROPOSED METHOD

Here we are presenting a second refinement of the GSOR Method. by using the equation $\tilde{y}^{(k+1)} = y^{(k+1)} + (T_m - \omega L_m)^{-1}(b - Ay^{(k+1)})\omega$ and Substituting value of $y^{(k+1)}$ form eq. (4) we get

$$\tilde{y}^{(k+1)} = [(T_m - \omega E_m)^{-1}(1-\omega)T_m + \omega F_m)]^2 y^{(k)} + [I + (T_m - \omega E_m)^{-1}(1-\omega)T_m + \omega F_m](T_m - \omega E_m)^{-1}b + (T_m - \omega E_m)^{-1}b\omega - (\omega A(T_m - \omega E_m)^{-1})[(T_m - \omega E_m)^{-1}(1-\omega)T_m + \omega F_m)]^2 y^{(k)} + [I + (T_m - \omega E_m)^{-1}(1-\omega)T_m + \omega F_m](T_m - \omega E_m)^{-1}b \quad (5)$$

After rearranging and simplifying eq. (5) we get

$$\tilde{y}^{(k+1)} = \left[(T_m - \omega E_m)^{-1}\left((1-\omega)T_m + \omega F_m\right)\right]^3 y^{(k)} + \left[I + (T_m - \omega E_m)^{-1}\left((1-\omega)T_m + \omega F_m\right) + \left((T_m - \omega E_m)^{-1}\left((1-\omega)T_m + \omega F_m\right)\right)^2\right](T_m - \omega E_m)^{-1}\,\omega b \quad (6)$$

Equation (6) is the equation of IIS. For $m = 0$ IIS becomes SRSOR.

6. CONVERGENCE THEORY

Theorem 1: Let A be an SDD matrix then for any vector $y^{(0)}$, the RGSOR converges.

Proof: (see [1])

Theorem 2: Let A be a square matrix of dimension $n \times n$, where $m \le n$ and m belongs to a set of natural numbers then the RGSOR converges for any vector $\mathbf{y}^{(0)}$.

Proof: (see [1])

Theorem 3: A is an SPD matrix and if $\omega < 2$ then SOR converges for any vector $\mathbf{y}^{(0)}$.

Proof: (see [6])

Theorem 4: Let A be a square matrix of dimension $n \times n$, where $m \le n$ and m belongs to a set of natural numbers then the IIS converges for any vector $\mathbf{y}^{(0)}$.

Proof: Suppose Y be the exact solution of equation 1. Let us assume that A be the SDD so SOR, GSOR and RGSOR will converge, i.e.

$$y^{(k+1)} \to y$$

When:

$$y^{(k+1)} = [(T_m - \omega L_m)^{-1}((1-\omega)T_m + \omega U_m)]^2 y^{(k)} + (T_m - \omega L_m)^{-1}[I + (T_m - \omega L_m)^{-1}((1-\omega)T_m + \omega U_m)]\omega b \quad (6.1)$$

Also,

$$y^{(k+1)} = y^{(k+1)} + (T_m - \omega L_m)^{-1}(b\omega - \omega A y^{(k+1)})$$

$$\tilde{y}^{(k+1)} - Y = y^{(k+1)} - Y + (T_m - \omega L_m)^{-1}\left(b\omega - \omega A y^{(k+1)}\right) \quad (6.2)$$

Taking norm on both sides:

$$\left\|\tilde{y}^{(k+1)} - Y\right\| = \left\|y^{(k+1)} - Y + (T_m - \omega L_m)^{-1}(b\omega - \omega A y^{(k+1)})\right\|$$

$$\le \left\|y^{(k+1)} - Y\right\| + \left\|(T_m - \omega L_m)^{-1}(b\omega - \omega A y^{(k+1)})\right\|$$

$$\therefore \left\|y^{(k+1)} - Y\right\| \le \left\|y^{(k+1)} - Y\right\| + \omega\left\|(T_m - \omega L_m)^{-1}\right\|\left\|\left(b - A y^{(k+1)}\right)\right\|$$

$$\left\|\tilde{y}^{(k+1)} - Y\right\| \le \|Y - \| + \omega\left\|(T_m - \omega L_m)^{-1}\right\|\|(b - b)\|$$

$$\left\|\tilde{y}^{(k+1)} - Y\right\| \le 0 + \omega\left\|(T_m - \omega L_m)^{-1}\right\| 0 = 0$$

Consequently $\left\|\tilde{y}^{(k+1)} - Y\right\| \to 0$

Then:

$$y^{(k+1)} \rightarrow Y$$

$$\Rightarrow \rho\left(\left[(T_m - \omega L_m)^{-1}\left((1-\omega)T_m + \omega U_m\right)\right]^3\right) = \left(\rho\left((T_m - \omega L_m)^{-1}\left((1-\omega)T_m + \omega U_m\right)\right)\right)^3 < 1 \quad (6.3)$$

Hence IIS is convergent.

Theorem 5: If be an $\boldsymbol{M}$-matrix of order $\boldsymbol{n} \times \boldsymbol{n}$ and m$\leq$ $\mathbf{n}$ where $\boldsymbol{m}$ belongs to a set of natural numbers then the IIS converges for any initial vector $\mathbf{y}^{(0)}$.

Proof:
We have an $\boldsymbol{M}$-matrix $\boldsymbol{A}$; we try to produce that IIS is convergent. using the convergent theorem of GSOR, we have:

$$\rho\left(T_m - \omega L_m)^{-1}\left((1-\omega)T_m + U_m\right)\right) < 1 \quad (6.4)$$

We realize that,

$\mathbf{G_{IIS}} = [\boldsymbol{\rho}(\mathbf{G_{GSOR}})]^3$Where $\boldsymbol{\rho}(\mathbf{G_{GSOR}})$ and $\mathbf{G_{IIS}}$ are spectral radii of GSOR and IIS respectively. As $\boldsymbol{\rho}(\mathbf{G_{GSOR}}) < \mathbf{1}$, so $\boldsymbol{\rho}(\mathbf{G_{IIS}}) < \boldsymbol{\rho}(\mathbf{G_{GSOR}}) < \mathbf{1}$

Theorem 6: The IIS method converges for any SPD matrix $\boldsymbol{A}$.

Proof:
Using theorem 3, We have $\boldsymbol{\rho}\left((T_m - \omega L_m)^{-1}[(1-\omega)T_m + \omega L_m]\right) < 1$.

Let X be the actual solution of eq. 1. Now by using eq. 2 we can write equation 1 as

$$Y = \left[I - (T_m - \omega L_m)^{-1}\left((1-\omega)T_m + \omega U_m\right)\right]^{-1} b\omega (T_m - \omega L_m)^{-1} \quad (7)$$

Using IIS

$$\tilde{y}^{(k+1)} = \left[(T_m - \omega L_m)^{-1}\left((1-\omega)T_m + \omega U_m\right)\right]^3 y^{(k)} + \left[I + (T_m - \omega L_m)^{-1}\left((1-\omega)T_m + \omega U_m\right) + \left((T_m - \omega L_m)^{-1}\left((1-\omega)T_m + \omega U_m\right)\right)^2\right](T_m - \omega L_m)^{-1}\,\omega b \quad (8)$$

Now using eq.(8) and the exact solution $\boldsymbol{Y}$, we have:

$$Y = \left[(T_m - \omega L_m)^{-1}\left((1-\omega)T_m + \omega U_m\right)\right]^3 y^{(k)} + \left[I + (T_m - \omega L_m)^{-1}\left((1-\omega)T_m + \omega U_m\right) + \left((T_m - \omega L_m)^{-1}\left((1-\omega)T_m + \omega U_m\right)\right)^2\right](T_m - \omega L_m)^{-1}\,\omega b$$

$$\Rightarrow Y = \left(1 - \left[(T_m - \omega L_m)^{-1}\left((1-\omega)T_m + \omega U_m\right)\right]^3\right)^{-1}\left[1 + (T_m - \omega L_m)^{-1}\left((1-\omega)T_m + \omega U_m\right) + \left((T_m - \omega L_m)^{-1}\left((1-\omega)T_m + \omega U_m\right)\right)^2\right](T_m - \omega U_m)^{-1}\omega b$$

$$Y = \left[1 + \left[(T_m - \omega L_m)^{-1}\left((1-\omega)T_m + \omega U_m\right)\right]^3 + \left[(1 - \omega L_m)^{-1}\left((1-\omega)T_m + \omega U_m\right)\right]^6 + \right]\left[I + (T_m - \omega L_m)^{-1}\left((1-\omega)T_m + \omega U_m\right) + \left((T_m - \omega L_m)^{-1}\left((1-\omega)T_m + \omega U_m\right)\right)^2\right](T_m - \omega U_m)^{-1}\omega b$$

Since $(1 - M)^{-1}$=1+M+M^2+…

If $\rho(M) < 1$ and 1-M is singular.

$$Y = \Big[1 + (T_m - \omega L_m)^{-1}\big((1-\omega)T_m + \omega U_m\big) + \Big((T_m - \omega L_m)^{-1}\big((1-\omega)T_m + \omega U_m\big)\Big)^2 + \Big((T_m - \omega L_m)^{-1}\big((1 - \omega)T_m + \omega U_m\big)\Big)^3 + \Big((T_m - \omega L_m)^{-1}\big((1 - \omega)T_m + \omega U_m\big)\Big)^3 + \cdots\Big](T_m - \omega U_m)^{-1}\omega b$$

$$Y = \big[1 - (T_m - \omega L_m)^{-1}((1-\omega)T_m + \omega U_m)\big]^{-1}(T_m - \omega U_m)^{-1}\omega b \quad (8.1)$$

As $(1 - M)^{-1} = 1 + M + M^2 + \cdots$

$\therefore Y = \big[1 - (T_m - \omega L_m)^{-1}((1-\omega)T_m + \omega U_m)\big]^{-1}(T_m - \omega U_m)^{-1}\omega b$ is consistent to GSOR.

Now examine the convergence of IIS for the SPD matrix.

$$\tilde{y}^{(k+1)} = [(T_m - \omega L_m)^{-1}((1-\omega)T_m + \omega U_m)]^{3} y^{(k)} + \Big[I + (T_m - \omega L_m)^{-1}\big((1-\omega)T_m + \omega U_m\big) + \Big((T_m - \omega L_m)^{-1}\big((1-\omega)T_m + \omega U_m\big)\Big)^2\Big](T_m - \omega L_m)^{-1}\omega b$$

$$y^{(k+1)} = [(T_m - \omega L_m)^{-1}((1-\omega)T_m + \omega U_m)]^{6} y^{(k-1)} + \Big[I + (T_m - \omega L_m)^{-1}\big((1-\omega)T_m + \omega U_m\big) + \Big((T_m - \omega L_m)^{-1}\big((1-\omega)T_m + \omega U_m\big)\Big)^2 + \Big((T_m - \omega L_m)^{-1}\big((1-\omega)T_m + \omega U_m\big)\Big)^3 + \Big((T_m - \omega L_m)^{-1}\big((1-\omega)T_m + \omega U_m\big)\Big)^4 + \Big((T_m - \omega L_m)^{-1}\big((1-\omega)T_m + \omega U_m\big)\Big)^5\Big](T_m - \omega L_m)^{-1}\omega b$$

$$\tilde{y}^{(k+1)} = [(T_m - \omega L_m)^{-1}((1-\omega)T_m + \omega U_m)]^{9} y^{(k-1)} + \Big[I + (T_m - \omega L)^{-1}\big((1-\omega)T_m + \omega U_m\big) + \Big((T_m - \omega L_m)^{-1}\big((1-\omega)T_m + \omega U_m\big)\Big)^2 + \Big((T_m - \omega L_m)^{-1}\big((1-\omega)T_m + \omega U_m\big)\Big)^3 + \Big((T_m - \omega L_m)^{-1}\big((1-\omega)T_m + \omega U_m\big)\Big)^4 + \Big((T_m - \omega L_m)^{-1}\big((1-\omega)T_m + \omega U_m\big)\Big)^5 + \Big((T_m - \omega L_m)^{-1}\big((1-\omega)T_m + \omega U_m\big)\Big)^6 + \Big((T_m - \omega L_m)^{-1}\big((1-\omega)T_m + \omega U_m\big)\Big)^7 + \Big((T_m - \omega L_m)^{-1}\big((1-\omega)T_m + \omega U_m\big)\Big)^8\Big](T_m - \omega L_m)^{-1}\omega b$$

$$y^{(k+1)} = [(T_m - \omega L_m)^{-1}((1-\omega)T_m + \omega U_m)]^{3k+3} y^{(0)} + \Big[I + (T_m - \omega L_m)^{-1}\big((1-\omega)T_m + \omega U_m\big) + \Big((T_m - \omega L_m)^{-1}\big((1-\omega)T_m + \omega U_m\big)\Big)^2 + \Big((T_m - \omega L_m)^{-1}\big((1-\omega)T_m$$

$$+\omega U_m)\Big)^3 + \cdots\cdots\Big((T_m-\omega L_m)^{-1}((1-\omega)T_m+\omega U_m)\Big)^{3k+2}\Big](T_m-\omega L_m)^{-1}\,\omega b \quad (8.2)$$

If A is SPD then:

$$\rho\left((T_m-\omega L_m)^{-1}((1-\omega)T_m+\omega U_m)\right)<1$$

$$\therefore \lim_{k\to\infty}\left[(T_m-\omega L_m)^{-1}((1-\omega)T_m+\omega U_m)\right]^{3k+3}=0$$

$$\Rightarrow \lim_{k\to\infty} y^{(k+1)} = \lim_{k\to\infty}\left[(T_m-\omega L_m)^{-1}((1-\omega)T_m+\omega U_m)\right]^{3k+3}\sum_{k=0}^{\infty}\left[(T_m-\omega L_m)^{-1}((1-\omega)T_m+\omega U_m)\right]^k (T_m-\omega L_m)^{-1}\,\omega b$$

$$=0+[I-(T_m-\omega L_m)^{-1}((1-\omega)T_m+\omega L_m)]^{-1}b\omega(T_m-\omega L_m)^{-1}\to Y$$

$$\Rightarrow \rho[((T_m-\omega L_m)^{-1}((1-\omega)T_m+\omega U_m))^3]=\left(\rho(T_m-\omega L_m)^{-1}((1-\omega)T_m+\omega U_m)\right)^3<1 \quad (8.3)$$

Theorem 7: If the SOR method converges then the IIS will converge more rapidly than GSOR and RGSOR.

Proof: The GSOR, RGSOR and IIS can be written respectively as,

$$y^{(k+1)}=Sy^{(k)}+I \quad (8.4)$$

$$y^{(k+1)}=S^2y^{(k)}+J \quad (8.5)$$

$$y^{(k+1)}=S^3x^{(k)}+K \quad (8.6)$$

Where $S=(T_m-\omega L_m)^{-1}[(1-\omega)T_m+\omega U_m)$

$$I=\omega(T_m-\omega L_m)^{-1}\mathrm{b}$$

$$J=[I+(T_m-\omega L_m)^{-1}((1-\omega)T_m+\omega U_m)](T_m-\omega L_m)^{-1}b,$$

$$\mathrm{K}=[I+(T_m-\omega L_m)^{-1}((1-\omega)T_m+\omega U_m)+((T_m-\omega L_m)^{-1}((1-\omega)T_m+\omega U_m))^2](T_m-\omega l_m)^{-1}\omega b$$

Let the exact solution of eq (1) is X

$\Rightarrow Y=SY+I,\ Y=S^2Y+J$ and $Y=S^3Y+K$

let $k=0,1,2,\ldots$ are nonnegative integer.

Let's consider the GSOR method

$$y^{(k+1)}=Sy^{(k)}+I$$

$$\Rightarrow y^{(k+1)}-Y=Sy^{(k)}-Y+I$$

$$\Rightarrow y^{(k+1)}-Y=Sy^{(k)}-Y+I+SY-SY$$

$$\Rightarrow y^{(k+1)}-Y=S(y^{(k)}-Y)+SY+I-Y$$

$$\Rightarrow y-Y=S(y^{(k)}-Y)+Y-Y$$

$$\Rightarrow y^{(k+1)}-Y=S(y^{(k)}-Y)$$

$$\Rightarrow \left\|y^{(k+1)}-Y\right\|=\left\|S(y^{(k)}-Y)\right\| \le \|S\|\left\|(y^{(k)}-Y)\right\| \le \left\|S^2\right\|\left\|(y^{(k-1)}-Y)\right\|\le\cdots \le \left\|S^k\right\|\left\|(y^{(1)}-Y)\right\|$$

$$\Rightarrow \left\|y^{(k+1)}-Y\right\|\le\left\|S^k\right\|\left\|(y^{(1)}-Y)\right\|\le \|S\|^k\left\|y^{(1)}-Y\right\| \quad \text{(a)}$$

Similarly, consider the refinement of GSOR and IIS.

$$\Rightarrow \left\|y^{(k+1)}-Y\right\|\le\left\|S^{2k}\right\|\left\|(y^{(1)}-Y)\right\|\le \|S\|^{2k}\left\|(y^{(1)}-Y)\right\| \quad \text{(b)}$$

and

$$\Rightarrow \left\|y^{(k+1)}-Y\right\|\le\left\|S^{3k}\right\|\left\|(y^{(1)}-Y)\right\|\le \|S\|^{3k}\left\|(y^{(1)}-Y)\right\| \quad \text{(c)}$$

Using the inequalities, a, b and c, since $\|S\|<1$ we conclude that the IIS converges faster than GSOR and RGSOR if SOR converges.

7. RESULTS AND DISCUSSION

In this section some numerical examples have been experimented with. The comparative analysis of the proposed scheme is done with GSOR and RGSOR by using spectral radius and stability.

The below example 1, 2, 3 and 4 are referred from previously conducted studies [6-13]. Also, the results are depicted in tables 1-4 and figure 1 and 2.

Example 1:

$$\boldsymbol{A}=\begin{bmatrix}5&-1&0&0&-1&0&0&-1\\-1&5&-1&0&0&0&-1&-1\\0&-1&5&-1&0&-1&-1&0\\1&0&-1&5&-1&0&0&-1\\-1&-1&0&0&5&-1&0&-1\\0&0&-1&-1&0&5&1&-1\\-1&0&0&0&-1&0&5&-1\\-1&0&-1&0&-1&0&-1&5\end{bmatrix};\boldsymbol{b}=\begin{bmatrix}-2\\-1\\4\\13\\4\\2\\9\\12\end{bmatrix}$$

Example 2:

$$\boldsymbol{A}=\begin{bmatrix}7&-1&0&-1&0&-1&0&-1&0&0\\-1&7&-1&0&-1&0&-1&0&-1&0\\0&-1&7&-1&0&-1&0&-1&0&-1\\1&0&-1&7&-1&0&-1&0&-1&0\\0&-1&0&-1&7&-1&0&-1&0&-1\\-1&0&-1&0&-1&7&-1&0&-1&0\\0&-1&0&-1&0&-1&7&-1&0&-1\\-1&0&-1&0&-1&0&-1&7&-1&0\\0&-1&0&-1&0&-1&0&-1&7&-1\\0&0&-1&0&-1&0&-1&0&-1&7\end{bmatrix};\boldsymbol{b}=\begin{bmatrix}7\\3\\2\\2\\2\\2\\2\\2\\2\\2\end{bmatrix}$$

Example 3:

$$A=\begin{bmatrix}-4&1&0&1&0&0&0&0&0\\1&-4&1&0&1&0&0&0&0\\0&1&-4&0&0&1&0&0&0\\1&0&0&-4&-1&0&0&0&-1\\0&1&0&1&-5&-1&0&0&-1\\0&0&1&0&1&-4&0&0&1\\0&0&0&1&0&0&-4&1&0\\0&0&0&0&1&0&1&-4&1\\0&0&0&0&0&1&0&1&-4\end{bmatrix};b=\begin{bmatrix}1/512\\4/512\\9/512\\4/512\\16/512\\36/512\\9/512\\36/512\\81/512\end{bmatrix}$$

Example 4:

$$A=\begin{bmatrix}2&-3&0&0&0&0\\-1&4&-1&0&-1&0\\0&-1&4&0&0&-1\\0&0&0&2&-3&0\\0&-1&0&-1&4&-1\\0&0&-1&0&-1&4\end{bmatrix};b=\begin{bmatrix}-5/3\\2/3\\3\\-4/3\\-1/3\\5/3\end{bmatrix}$$

Fig. 1. The spectral radius of the improved method in comparison to other schemes.

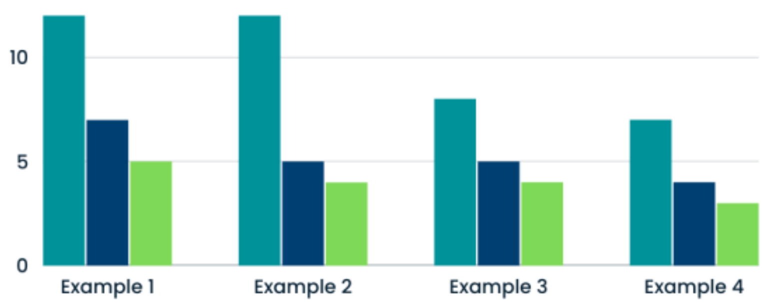

Fig. 2. The number of iterations taken by the improved method in comparison to other schemes.

Table 1. Numerical Results from example 1.

Methods	No of iterations	Spectral Radius	Stability (infinity norm) set up to 4th iteration
GSOR	12	0.1319	0.0372
RGSOR	07	0.0174	0.0004
IIS	05	0.0022	0.000003

Table 2. Numerical Results from example 2.

Methods	No of iterations	Spectral Radius	Stability (infinity norm) set up to 4th iteration
GSOR	12	0.1460	0.02777
RGSOR	05	0.0350	0.00024
IIS	04	0.0066	0.000009

Table 3. Numerical Results from example 3.

Methods	No of iterations	Spectral Radius	Stability (infinity norm) set up to 4th iteration
GSOR	08	0.1617	0.00099
RGSOR	05	0.0261	0.000013
IIS	04	0.0042	0.00000044

Table 4. Numerical Results from example 4.

Methods	No of iterations	Spectral Radius	Stability (infinity norm) set up to 4th iteration
GSOR	07	0.5954	0.0589
RGSOR	04	0.3546	0.0005
IIS	03	0.2111	0.000003

The null vector is used as an initial approximation with a tolerance of 0.00001. The value of ω is taken optimally. In example 01, The coefficient matrix A is SDD and SPD with m=1 and ω=1.0695. In example 02, A is SDD and an M-matrix with m=1 and ω=1.099. In the example, 03 A is an SDD matrix with m=1 and ω=1.1617. In example 4, A is an M matrix with m=1 and ω=1.098.

8. CONCLUSION

An improved iterative scheme using successive over-relaxation for the solution of a linear system of equations is presented in this paper. The convergence of IIS for M-matrix, SPD and SDD, is examined and four numerical examples are presented using MATLAB version R2014b (8.4.0.150421). In aspects of the number of iterations and error analysis, all results obtained by IIS are compared to the first refinement of generalized SOR and generalized SOR, as shown in tables 1, 2, 3 and 4. The evolution of the result shows that the IIS converges faster than the GSOR and RGSOR.

The presented method works efficiently, however, it is only applicable to M-matrix, SPD and SDD matrices, Future work can be done in the application of similar improvement techniques to methods that are more robust and can handle a large variety of matrices.

9. CONFLICT OF INTEREST

The authors declare no conflict of interest.

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