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Research Article

Evaluation of Electric Field for a Dielectric Cylinder Placed in Fractional Space

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Abstract: The problem related to the dielectric cylinder placed in non-integer dimensional space (FD space) is thoroughly investigated in this paper. The FD space describes complex phenomena of physics and electromagnetism. We have solved Laplacian equation in FD space to obtain the solution of a dielectric cylinder in low frequency. The problem is solved by the method of separation of variables analytically. The classical solution of the problem can be easily recovered from the derived solution in non-integer dimensional space.

Keywords: FD-Space,--Laplacian-Equation,--Quantum-Field-Theory,--Ising-limit

1. INTRODUCTION

The concept of non-integer dimensional space (FD -space) has been considered a very useful in various areas of physics and electromagnetism and many researchers [1 – 16] have discussed and applied it previously. Wilson [3] has employed this idea in the quantum field theory. Further, it was suggested that the FD-space can be used as a parameter in the Ising limit in quantum field theory [6]. Stillinger [4] has provided an axiomatic basis of this concept for the formulation of Schrodinger wave mechanics and Gibbsian statistical mechanics in α -dimensional space. Svozil and Zeilinger [10] have presented operationalized definition of the of space–time which has provided the possibility of experimental determination of space–time dimension. It has also been stated that the non-integer dimension of space–time is slightly less than 4. Gauss law [11] has been formulated in α-dimensional fractional space. The solution of electrostatic problems [13 − 18] have also been investigated in the FD space considering $(2 < \alpha \leq 3)$. In the present work, we have extended the dielectric cylinder problem and solved it analytically in non-integer dimensional space. The main objective is to use the Laplacian equation to find electric potential and the field due to a dielectric cylinder in non-integer dimensional space. To retrieve the integer order, we may consider $\alpha = 3$. As a result, the original solution is recovered.

2. MATHEMATICAL MODEL

We have considered an infinitely long circular cylinder of radius 'a', which is made up of a material having a dielectric constant (ϵ/ϵ_0) and is placed in uniform electric field E0. The cylinder is oriented with its axis at the right angle to the applied primary field $E0r(\alpha-2)\cos\theta$. We will find the potential and electric field in non-integer space ($2 < \alpha \leq 3$) in the three regions. We will employ the cylindrical coordinates (r,θ) for the appropriate solutions.

Since the total charge enclosed within the region is zero, so we can use Poisson's equation:

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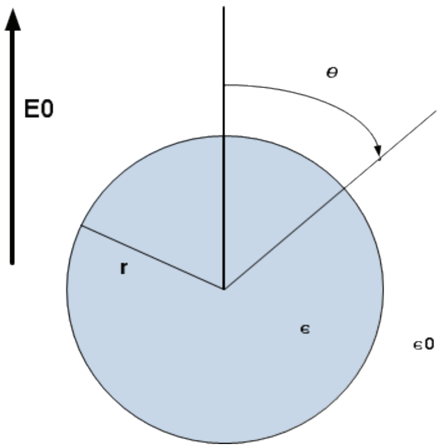

Fig.1: Dielectric Cylinder Placed in Fractional Space

Since the total charge enclosed within the region is zero, so we can use Poisson's equation:

$$\nabla^2\Phi(r,\theta) = 0 \quad (1)$$

This is also known as cylindrical wave equation and is expressed as follows,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial}\right) + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\phi^2} + \frac{\partial^2\Psi}{\partial z^2} + k^2 \Phi = 0 \quad (2)$$

We will deal with this problem in electrostatic and magnetostatics, where $\omega = 0$ so that $k = 0$. As the translational symmetry of the cylinder is considered to be along z-axis, so 'Φ' is independent of 'z' and we need to consider the problem in the (r,θ)- plane only. Further, symmetry in this problem leads us to choose cylindrical coordinates in which Poisson's equation is expressed as follows,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = 0 \quad (3)$$

The separation of variable method solves the Eq. (3) and its possible solutions in the uniform electric field are $r\cos\theta$ and $r^{-1}\cos\theta$. The general solution for low frequency can be expressed as follows,

$$\Psi(r,\phi) = \sum_{l=0}^{\infty}\left(A_l r^l + B_l r^{-l}\right) P_l(\cos\theta \quad (4)$$

where, $P_1(\cos\theta) = \cos\theta$.

Eq.(3) can also be solved by separable method in non-integer space.

Let suppose,

$$\varphi(r,\theta) = R(r)\Theta(\theta) \quad (5)$$

$$\left[\frac{d^2}{d\theta^2} + (\alpha-2)\cot\theta\frac{d}{d\theta} + l(l+\alpha-2)\right]\Theta(\theta) = 0 \quad (6)$$

(6)

$$\left[\frac{d^2}{dr^2} + \frac{\alpha-2}{r}\frac{d}{dr} + \frac{l(l+\alpha-3)}{r^2}\right]R(r) = 0 \quad (7)$$

The solutions of the above angular differential equation (6) is obtained from [10] and expressed as Follows,

$$\Theta(\theta) = P_l^{\alpha/2-1}(\cos\theta) \quad (8)$$

(1)

Similarly, the solutions of the above radial differential equation (7) is obtained from [12] as r^l and $r^{-(l+\alpha-3)}$. Therefore, the general solution of scalar potential of dielectric cylinder in fraction space can be expressed as,

$$\Psi_e(r,\varphi) = \sum_{l=0}\left\langle a_l r^l + b_l r^{-(l+\alpha-3)}\right\rangle P_l^{\alpha/2-1}(\cos\theta) \quad (9)$$

(2)

For our convenience, we can limit the above form of the solution only within outside and inside of the cylindrical regions. For the outside region, we need to have the electric field at infinity, but we certainly don't want the field to diverge. It is that the logarithmic and r^l terms with $l > 1$ diverge as 'r' goes to infinity. Hence , these terms are unphysical and can not be considered. Therefore, we are interested only in the solution, for $l = 1$

$$P_1^{\alpha/2-1}(\cos\theta) = (\alpha-2)\cos\theta.$$

Because each region has the same symmetry with respect to the external field, so the expressions of potentials in each region are written as,

Outside the region:

$$\Psi(r,\phi) = -E_0 r(\alpha-2)\cos\phi + \frac{B}{r^{\alpha-2}}(\alpha-2)\cos\phi, \quad r > a \quad (10)$$

Here, for large values of 'r' the field is supposed to reduce to $-E_0 r(\alpha-2)\cos\varphi$, corresponding to the uniform field.

Inside the region:

$$\Psi^i(r,\varphi) = A\, r(\alpha - 2) \cos\varphi \qquad r < a \quad (11)$$

For the determination of unknown coefficients 'A' and 'B' boundary conditions are applied. The fields must be continuous across the boundary at $r = a$, so using the boundary conditions we find the unknown coefficients A and B, which are expressed as follows.

$$A = \frac{-(\alpha-1)E_0}{\epsilon + (\alpha - 2)} \qquad (12)$$

$$B = E_0 a^{(\alpha-1)} \frac{\epsilon - 1}{\epsilon + (\alpha - 2)} \qquad (13)$$

$$\Psi = -E_0 r(\alpha-2)\cos\phi + E_0\left(\frac{\epsilon-1}{\epsilon+(\alpha-2)}\right)\frac{a^{\alpha-1}}{r^{\alpha-1}}(\alpha-2)\cos\phi, \quad r > a \qquad (14)$$

$$\Psi^i(r,\phi) = \frac{-(\alpha-1)E_0}{\epsilon+(\alpha-2)} r(\alpha-2)\cos\phi \quad r < a \qquad (15)$$

The total electric field intensity is

$$E^t = E_0(\alpha-2)\left(1 + a^{\alpha-1}\left(\frac{\epsilon-1}{\epsilon+(\alpha-2)}\right)\right)\left[\frac{(x^2-z^2)}{r^{\alpha+1}}u_x + \frac{(\alpha-1)xz}{r^{\alpha+1}}u_z\right] \qquad (16)$$

The secondary electric field intensity is

$$E^p = E_0(\alpha-2)a^{\alpha-1}\left(\frac{\epsilon-1}{\epsilon+(\alpha-2)}\right)\left[\frac{(x^2-z^2)}{r^{\alpha+1}}u_x + \frac{(\alpha-1)xz}{r^{\alpha+1}}u_z\right] \qquad (17)$$

3. CONCLUSION

In this paper the Laplace equation has been studied in α -dimensional fractional space. The potential and electric field of the dielectric cylinder is obtained in fractional space. The classical results are recovered from the investigated solution for $\alpha = 3$. Further, this solution can be applied for various materials. The host medium and core medium can be studied for multiple materials like meta-materials, plasma etc.

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