Proceedings of the Pakistan Academy of Sciences: A: Physical and Computational Sciences 59(1): 29-36 (2022)

ISSN (Print): 2518-4245; ISSN (Online): 2518-4253
https://doi.org/10.53560/PPASA(59-1)640

Pakistan Academy of Sciences

Research Article

Class of Meromorphic Univalent Functions with Fixed Second Positive Coefficients Defined by q-Difference Operator

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Abstract: In this paper using a q-difference operator, a class of meromorphic univalent functions with fixed second positive coefficients is defined. Coefficient estimates, some distortion theorems and other properties for this class are obtained. Various results obtained are sharp.

Keywords: Meromorphic, starlike, convex, fixed coefficient, radius of convexity.

1. INTRODUCTION

For $0 \leq \delta < 1$, let Σ_δ denote the class of univalent meromorphic functions of the form:

$$F(\varsigma) = \frac{1}{\varsigma-\delta} + \sum_{k=1}^{\infty} a_k \varsigma^k, F(\delta) = \infty,$$

defined in the desk $\mathcal{D}_\delta = \{\varsigma : \delta < |\varsigma| < 1\}$.
Also let $\Sigma_{\delta,\alpha}$, $0 < \alpha \leq 1$ be the subclass of functions F in Σ_δ which has the expansion:

$$F(\varsigma) = \frac{\alpha}{\varsigma-\delta} + \sum_{k=1}^{\infty} a_k \varsigma^k, \quad (1.1)$$

where $\alpha = Res(F, \delta)$, with $0 < \alpha \leq 1$, $\varsigma \in \mathcal{D}_\delta$.
The function F given in (1.1) was studied by Jinxi Ma [12]. The functions $F \in \Sigma_\delta$ is said to be meromorphically starlike (convex) functions of order β if and only if

$$-Re\left\{\frac{\varsigma F'(\varsigma)}{F(\varsigma)}\right\} > \beta, 0 \leq \beta < 1, \varsigma \in \mathcal{D}_\delta, \quad (1.2)$$

$$-Re\left\{1 + \frac{\varsigma F''(\varsigma)}{F'(\varsigma)}\right\} > \beta, 0 \leq \beta < 1, \varsigma \in \mathcal{D}_\delta. \quad (1.3)$$

The class of such functions is denoted by $\Sigma_\delta^*(\beta)$ $(\Sigma_\delta^c(\beta))$. Note that the class $\Sigma_0^*(\beta)$ and various other subclasses of $\Sigma_\delta^*(0)$ had been studied by [5](see also [1, 2], [10], [13], [15], [17, 18, 19]).

Let $\Sigma_{\delta,\alpha}^+ \subset \Sigma_{\delta,\alpha}$ consisting of functions of the form:

$$F(\varsigma) = \frac{\alpha}{\varsigma-\delta} + \sum_{k=1}^{\infty} a_k \varsigma^k, (a_k \geq 0). \quad (1.4)$$

It is known that the calculus without the notion of limits is called $q-$calculus which has influenced many scientific fields due to its important applications. Tang et al. [16] defined the $q-$derivative for meroromorphic functions $F \in \Sigma_0$ by:

$$\partial_q F(\varsigma) = \frac{F(\varsigma)-F(q\varsigma)}{(1-q)\varsigma} = -\frac{1}{q\varsigma^2} + \sum_{k=1}^{\infty} [k]_q a_k \varsigma^{k-1}, \quad (1.5)$$

where

$$[j]_q = \frac{1-q^j}{1-q}. \quad (1.6)$$

As $q \to 1^-$, $[j]_q = j$ and $\partial_q F(\varsigma) = F'(\varsigma)$.

For $F \in \Sigma_{\delta,\alpha}$, let:

$$\mathcal{M}_q^0 F(\varsigma) = F(\varsigma),$$

$$\mathcal{M}_q^1 F(\varsigma) = \varsigma \partial_q F(\varsigma) + \frac{\alpha((q+1)\varsigma - \delta)}{(\varsigma-\delta)(q\varsigma-\delta)},$$

$$\mathcal{M}_q^2 F(\varsigma) = \varsigma \partial_q (\mathcal{M}_q^1 F(\varsigma)) + \frac{\alpha((q+1)\varsigma - \delta)}{(\varsigma-\delta)(q\varsigma-\delta)},$$

Received: November 2021; Accepted: March 2022
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and for $n \in \mathbb{N} = \{1,2,3,\dots\}$ we can write

$$\mathcal{M}_q^n F(\varsigma) = \varsigma\partial_q(\mathcal{M}_q^{n-1}F(\varsigma)) + \frac{\alpha((q+1)\varsigma-\delta)}{(\varsigma-\delta)(q\varsigma-\delta)}$$
$$= \frac{\alpha}{\varsigma-\delta} + \sum_{k=1}^{\infty} [k]_q^n a_k \varsigma^k. \qquad (1.7)$$

Note that:

(i) $\lim_{q\to 1^-} \mathcal{M}_q^n(\delta,\alpha) = \mathcal{M}^n(\delta,\alpha)$ (see [7, 8, 9]);

(ii) $\lim_{q\to 1^-} \mathcal{M}_q^n(0,1) = \mathcal{M}^n$ (see [6]).

Using the operator $\mathcal{M}_q^n$, and for $F \in \Sigma_{\delta,\alpha}$ we have:

Definition 1 *The function $F \in \Sigma_q^n(\delta,\alpha,\beta)$ if it satisfies*

$$\left|\frac{\varsigma q\partial_q\left(\mathcal{M}_q^n F(\varsigma)\right)}{\mathcal{M}_q^n F(\varsigma)} + 1\right| < \left|\frac{\varsigma q\partial_q\left(\mathcal{M}_q^n F(\varsigma)\right)}{\mathcal{M}_q^n F(\varsigma)} + 2\beta - 1\right| \; (n \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}), \qquad (1.8)$$

for some $\beta (0 \le \beta < 1)$.

For $q \to 1^-$, $\Sigma_q^0(\delta,1,\beta)$ is the class of meromorphically starlike functions of order β and $\Sigma_q^0(\delta,1,0)$ gives the meromorphically starlike functions for all $\varsigma \in \mathcal{D}_\delta$.

Note that:

i. $\lim_{q\to 1^-} \Sigma_q^n(\delta,\alpha,\beta) = \Sigma^n(\delta,\alpha,\beta)$

$$= \left\{F(\varsigma): \left|\frac{\varsigma(\mathcal{M}^n F(\varsigma))'}{\mathcal{M}^n F(\varsigma)} + 1\right| < \left|\frac{\varsigma(\mathcal{M}^n F(\varsigma))'}{\mathcal{M}^n F(\varsigma)} + 2\beta - 1\right|\right\}$$

(see [9]);

ii. $\Sigma_q^n(0,1,\beta) = \Sigma_q^n(\beta)$

$$= \left\{F(\varsigma): \left|\frac{\varsigma q\partial_q(\mathcal{D}^n F(\varsigma))}{\mathcal{D}^n F(\varsigma)} + 1\right| < \left|\frac{\varsigma q\partial_q(\mathcal{D}^n F(\varsigma))}{\mathcal{D}^n F(\varsigma)} + 2\beta - 1\right|\right\};$$

iii. $\lim_{q\to 1^-} \Sigma_q^n(0,1,\beta) = \Sigma^n(\beta)$

$$= \left\{F(\varsigma): \left|\frac{\varsigma(\mathcal{D}^n F(\varsigma))'}{\mathcal{D}^n F(\varsigma)} + 1\right| < \left|\frac{\varsigma(\mathcal{D}^n F(\varsigma))'}{\mathcal{D}^n F(\varsigma)} + 2\beta - 1\right|\right\}.$$

Let $\Sigma_q^n[\delta,\alpha,\beta] = \Sigma_q^0(\delta,\alpha,\beta) \cap \Sigma_{\delta,\alpha}^+$, where $\Sigma_{\delta,\alpha}^+$ is the class of functions of the form (1.4) that are analytic and univalent in $\mathcal{D}_\delta$.

Following Goodman [11] and Ruscheweyh [14], we begin by introducing here the N_δ −neighborhood for $F(\varsigma) \in \Sigma_\delta$ by

$$N_\delta(F,g) = \{g: g(\varsigma) \in \Sigma_\delta, g(\varsigma) = \frac{1}{\varsigma} + \sum_{k=1}^{\infty} b_k\varsigma^k \text{ and } \sum_{k=1}^{\infty} k|a_k - b_k| \le \delta\},$$

and for $e(\varsigma) = \frac{1}{\varsigma}$;

$$N_\delta(e,g) = \{g: g(\varsigma) \in \Sigma_\delta, g(\varsigma) = \frac{1}{\varsigma} + \sum_{k=1}^{\infty} b_k\varsigma^k \text{ and } \sum_{k=1}^{\infty} k|b_k| \le \delta\}.$$

In [4] Aouf et al. (see also Madian and Aouf [3] (with $p = 1$)) defined the $N_{q,\delta}$ −neighborhood for $F(\varsigma) \in \Sigma_\delta$ by

$$N_{q,\delta}(F,g) = \{g: g(\varsigma) \in \Sigma_\delta, g(\varsigma) = \frac{1}{\varsigma} + \sum_{k=1}^{\infty} b_k\varsigma^k \text{ and } \sum_{k=1}^{\infty} [k]_q|a_k - b_k| \le \delta_q\}, \qquad (1.9)$$

and for $e(\varsigma) = \frac{1}{\varsigma}$;

$$N_{q,\delta}(e,g) = \{g: g(\varsigma) \in \Sigma_\delta, g(\varsigma) = \frac{1}{\varsigma} + \sum_{k=1}^{\infty} b_k\varsigma^k \text{ and } \sum_{k=1}^{\infty} [k]_q|b_k| \le \delta_q\}. \qquad (1.10)$$

2 MAIN RESULTS

Unless indicated, let $0 < q < 1$, $n \in \mathbb{N}_0$, $0 < \alpha \le 1$, $0 \le \beta < 1$, $\varsigma \in \mathcal{D}_\delta$.

Theorem 1 *Let $F(\varsigma)$ be defined by (1.4). Then $F \in \Sigma_q^n[\delta,\alpha,\beta]$ if and only if*

$$\sum_{k=1}^{\infty} [k]_q^n (\mathrm{q}[k]_q + \beta)(1-\delta)a_k \le \alpha(1-\beta). \qquad (2.1)$$

Proof. Assume that (2.1) holds true and let $|\varsigma| = 1$, by (1.8) we get

$$\left|\frac{\varsigma q\partial_q(\mathcal{M}_q^n F(\varsigma))}{\mathcal{M}_q^n F(\varsigma)} + 1\right| - \left|\frac{\varsigma q\partial_q(\mathcal{M}_q^n F(\varsigma))}{\mathcal{M}_q^n F(\varsigma)} + 2\beta - 1\right|$$
$$\le \frac{-2\alpha(1-\beta)}{|\varsigma||\varsigma-\delta|} - 2\sum_{k=1}^{\infty} [k]_q^n(\mathrm{q}[k]_q + \beta)a_k|\varsigma|^{k-1}$$
$$\le \frac{-2\alpha(1-\beta)}{1-\delta} - 2\sum_{k=1}^{\infty} [k]_q^n(\mathrm{q}[k]_q + \beta)a_k < 0.$$

We have

$$\sum_{k=1}^{\infty} 2[k]_q^n(\mathrm{q}[k]_q+\beta)(1-\delta)a_k - 2\alpha(1-\beta) < 0$$

Therefore, by the maximum modules theorem, we have $F \in \Sigma_q^n[\delta,\alpha,\beta]$.

Now, let $F \in \Sigma_q^n[\delta,\alpha,\beta]$, then

$$\left|\frac{\dfrac{\varsigma q\partial_q(\mathcal{M}_q^nF(\varsigma))}{\mathcal{M}_q^nF(\varsigma)}+1}{\dfrac{\varsigma q\partial_q(\mathcal{M}_q^nF(\varsigma))}{\mathcal{M}_q^nF(\varsigma)}+2\beta-1}\right|<1,$$

since $Re(\varsigma) \le |\varsigma|$ for all ς, we get

$$Re\left\{\frac{\frac{-\alpha\delta}{\varsigma(\varsigma-\delta)(q\varsigma-\delta)}+\sum_{k=1}^{\infty}[k]_q^n(\mathrm{q}[k]_q+1)a_k\varsigma^{k-1}}{\frac{\alpha(2\beta-1)}{\varsigma(\varsigma-\delta)}-\frac{\alpha q}{(\varsigma-\delta)(q\varsigma-\delta)}+\sum_{k=1}^{\infty}[k]_q^n(\mathrm{q}[k]_q+2\beta-1)a_k\varsigma^{k-1}}\right\}<1.$$

Choose values of ς on real axis so that $\frac{\varsigma q\partial_q(\mathcal{M}_q^nF(\varsigma))}{\mathcal{M}_q^nF(\varsigma)}$ is real. Letting $\varsigma \to 1^-$ through real values, we have (2.1).

Corollary 1 *If* $F \in \Sigma_q^n[\delta,\alpha,\beta]$, *then we have*

$$a_k \le \frac{\alpha(1-\beta)}{[k]_q^n(\mathrm{q}[k]_q+\beta)(1-\delta)}. \tag{2.2}$$

Equality is attained for the function F:

$$F(\varsigma)=\frac{\alpha}{\varsigma-\delta}+\frac{\alpha(1-\beta)}{[k]_q^n(\mathrm{q}[k]_q+\beta)(1-\delta)}\varsigma^k \tag{2.3}$$

Let $\Sigma_q^n[\delta,\alpha,\beta,c] \subset \Sigma_q^n[\delta,\alpha,\beta]$ consisting of functions:

$$F(\varsigma)=\frac{\alpha}{\varsigma-\delta}+\frac{\alpha(1-\beta)c}{(\mathrm{q}+\beta)(1-\delta)}\varsigma+\sum_{k=2}^{\infty}a_k\varsigma^k, \tag{2.4}$$

with $0 \le c < 1$.

Theorem 2 *Let* $F(\varsigma)$ *be defined by (2.4). Then* $F(\varsigma) \in \Sigma_q^n[\delta,\alpha,\beta,c]$ *if and only if*

$$\sum_{k=2}^{\infty}[k]_q^n(\mathrm{q}[k]_q+\beta)(1-\delta)a_k \le \alpha(1-\beta)(1-c). \tag{2.5}$$

Proof. Putting

$$a_1=\frac{\alpha(1-\beta)c}{(\mathrm{q}+\beta)(1-\delta)} \quad (0<c<1), \tag{2.6}$$

in (2.1), we have

$$c_1+\sum_{k=2}^{\infty}\frac{[k]_q^n(\mathrm{q}[k]_q+\beta)(1-\delta)}{\alpha(1-\beta)}a_k \le 1, \tag{2.7}$$

which implies (2.5). The equality accurs for

$$F(\varsigma)=\frac{\alpha}{\varsigma-\delta}+\frac{\alpha(1-\beta)c}{(\mathrm{q}+\beta)(1-\delta)}\varsigma+\frac{\alpha(1-\beta)(1-c)}{[k]_q^n(\mathrm{q}[k]_q+\beta)(1-\delta)}\varsigma^k, \tag{2.8}$$

for $k \ge 2$.

Corollary 2 *If* $F(\varsigma) \in \Sigma_q^n[\delta,\alpha,\beta,c]$, *then*

$$a_k \le \frac{\alpha(1-\beta)(1-c)}{[k]_q^n(\mathrm{q}[k]_q+\beta)(1-\delta)}, \ (k \ge 2). \tag{2.9}$$

The equality accurs for $F(\varsigma)$ given by (2.8).

Theorem 3 *If* $F \in \Sigma_q^n[\delta,\alpha,\beta,c]$, *then*

$$\sum_{k=2}^{\infty}a_k \le \frac{\alpha(1-\beta)(1-c)}{[2]_q^n(\mathrm{q}[2]_q+\beta)(1-\delta)}, \tag{2.10}$$

and

$$\sum_{k=2}^{\infty}[k]_qa_k \le \frac{[2]_q\alpha(1-\beta)(1-c)}{[2]_q^n(\mathrm{q}[2]_q+\beta)(1-\delta)}. \tag{2.11}$$

Proof. Let $F \in \Sigma_q^n[\delta,\alpha,\beta,c]$. Then, in view of (2.5), we have

$$[2]_q^n(\mathrm{q}[2]_q+\beta)(1-\delta)\sum_{k=2}^{\infty}a_k \le \alpha(1-\beta)(1-c), \tag{2.12}$$

which immediately yields the first assertior

By appealing to (2.5), we have

$$[2]_q^n(1-\delta)\sum_{k=2}^{\infty}\mathrm{q}[k]_qa_k \le \alpha(1-\beta)(1-c)-\beta[2]_q^n(1-\delta)\sum_{k=2}^{\infty}a_k, \tag{2.13}$$

which in view of (2.10), can be putten in the form:

$$[2]_q^n(1-\delta)\sum_{k=2}^{\infty}\mathrm{q}[k]_qa_k \le \alpha(1-\beta)(1-c)-\beta\frac{\alpha(1-\beta)(1-c)}{(\mathrm{q}[2]_q+\beta)}. \tag{2.14}$$

Simplifying the right hand side of (2.14), we have (2.11).

Theorem 4 *Let* $F(\varsigma) \in \Sigma_q^n[\delta,\alpha,\beta,c]$ *for* $0<|\varsigma|=r<1$.

Then

$$\frac{\alpha}{|\varsigma-\delta|}-\frac{\alpha(1-\beta)c}{(\mathrm{q}+\beta)(1-\delta)}|\varsigma|-\frac{\alpha(1-\beta)(1-c)}{[2]_q^n(\mathrm{q}[2]_q+\beta)(1-\delta)}|\varsigma|^2$$

$$\leq |F(\varsigma)| \leq \frac{\alpha}{|\varsigma-\delta|}+\frac{\alpha(1-\beta)c}{(\mathrm{q}+\beta)(1-\delta)}|\varsigma| + \frac{\alpha(1-\beta)(1-c)}{[2]_q^n(\mathrm{q}[2]_q+\beta)(1-\delta)}|\varsigma|^2, \tag{2.15}$$

with equality for

$$F(\varsigma)=\frac{\alpha}{|\varsigma-\delta|}+\frac{\alpha(1-\beta)c}{(\mathrm{q}+\beta)(1-\delta)}|\varsigma| + \frac{\alpha(1-\beta)(1-c)}{[2]_q^n(\mathrm{q}[2]_q+\beta)(1-\delta)}|\varsigma|^2,$$

where $\alpha = Res(\varsigma,\delta)$, with $0<c<1$.

Proof. For $F(\varsigma)\in\Sigma_q^n[\delta,\alpha,\beta,c]$. Then

$$|F(\varsigma)|=\left|\frac{\alpha}{\varsigma-\delta}+\frac{\alpha(1-\beta)c}{(\mathrm{q}+\beta)(1-\delta)}\varsigma+\sum_{k=2}^{\infty}a_k\varsigma^k\right|$$

$$\leq \frac{\alpha}{|\varsigma-\delta|}+\frac{\alpha(1-\beta)c}{(\mathrm{q}+\beta)(1-\delta)}|\varsigma|+|\varsigma|^2\sum_{k=2}^{\infty}a_k,$$

and

$$|F(\varsigma)|=\left|\frac{\alpha}{\varsigma-\delta}+\frac{\alpha(1-\beta)c}{(\mathrm{q}+\beta)(1-\delta)}\varsigma+\sum_{k=2}^{\infty}a_k\varsigma^k\right|$$

$$\geq \frac{\alpha}{|\varsigma-\delta|}-\frac{\alpha(1-\beta)c}{(\mathrm{q}+\beta)(1-\delta)}|\varsigma|-|\varsigma|^2\sum_{k=2}^{\infty}a_k,$$

which in view of (2.10), we have (2.15).

Theorem 5 *Let* $F(\varsigma)\in\Sigma_q^n[\delta,\alpha,\beta,c]$ *for* $0<|\varsigma|=r<1$, *then*

$$\frac{\alpha}{|\varsigma-\delta||q\varsigma-\delta|}-\frac{\alpha(1-\beta)c}{(\mathrm{q}+\beta)(1-\delta)} - \frac{\alpha(1-\beta)(1-c)}{[2]_q^{n-1}(\mathrm{q}[2]_q+\beta)(1-\delta)}|\varsigma|$$

$$\leq |\partial_q F(\varsigma)| \leq \frac{\alpha}{|\varsigma-\delta||q\varsigma-\delta|}+\frac{\alpha(1-\beta)c}{(\mathrm{q}+\beta)(1-\delta)} + \frac{\alpha(1-\beta)(1-c)}{[2]_q^{n-1}(\mathrm{q}[2]_q+\beta)(1-\delta)}|\varsigma|,$$

with equality for

$$\partial_q F(\varsigma)=\frac{\alpha}{|\varsigma-\delta||q\varsigma-\delta|}+\frac{\alpha(1-\beta)c}{(\mathrm{q}+\beta)(1-\delta)} + \frac{\alpha(1-\beta)(1-c)}{[2]_q^{n-1}(\mathrm{q}[2]_q+\beta)(1-\delta)}|\varsigma|.$$

Proof. For $F(\varsigma)\in\Sigma_q^n[\delta,\alpha,\beta,c]$. Then

$$|\partial_q F(\varsigma)|=\left|\frac{-\alpha}{(\varsigma-\delta)(\mathrm{q}\varsigma-\delta)}+\frac{\alpha(1-\beta)c}{(\mathrm{q}+\beta)(1-\delta)}+\sum_{k=2}^{\infty}[k]_q a_k\varsigma^{k-1}\right|$$

$$\leq \frac{\alpha}{|\varsigma-\delta||q\varsigma-\delta|}+\frac{\alpha(1-\beta)c}{(\mathrm{q}+\beta)(1-\delta)}+|\varsigma|\sum_{k=2}^{\infty}[k]_q a_k,$$

and

$$|\partial_q F(\varsigma)|=\left|\frac{-\alpha}{(\varsigma-\delta)(\mathrm{q}\varsigma-\delta)}+\frac{\alpha(1-\beta)c}{(\mathrm{q}+\beta)(1-\delta)}+\sum_{k=2}^{\infty}[k]_q a_k\varsigma^{k-1}\right|$$

$$\geq \frac{\alpha}{|\varsigma-\delta||q\varsigma-\delta|}-\frac{\alpha(1-\beta)c}{(\mathrm{q}+\beta)(1-\delta)}-|\varsigma|\sum_{k=2}^{\infty}[k]_q a_k,$$

which in view of (2.11), we have the result.

Theorem 6 *Let* $F(\varsigma)\in\Sigma_q^n[\delta,\alpha,\beta,c]$. *Then* $F(\varsigma)$ *is starlike of order* ν $(0\leq\nu<1)$ *in* $|\varsigma-\delta|<|\varsigma|<r_1$, *where* r_1 *is the largest value for which*

$$\frac{\alpha(3-\nu)(1-\beta)c}{(\mathrm{q}+\beta)(1-\delta)}r^2+\frac{\alpha(k_0+2-\nu)(1-\beta)(1-c)}{[k_0]_q^n(\mathrm{q}[k_0]_q+\beta)(1-\delta)}r^{k+1} \leq \alpha(1-\nu), \tag{2.16}$$

for $k\geq 2$. The result is sharp for the function $F(\varsigma)$ given by (2.8).

Proof. It is sufficent to show that

$$\left|\frac{(\varsigma-\delta)F'(\varsigma)}{F(\varsigma)}+1\right|\leq 1-\nu,\quad (|\varsigma|<r_1). \tag{2.17}$$

We have

$$\left|\frac{(\varsigma-\delta)F'(\varsigma)}{F(\varsigma)}+1\right| \leq \frac{\frac{2\alpha(1-\beta)c}{(\mathrm{q}+\beta)(1-\delta)}|\varsigma|+\sum_{k=2}^{\infty}(k+1)a_k|\varsigma|^k}{\frac{\alpha}{|\varsigma-\delta|}-\frac{\alpha(1-\beta)c}{(\mathrm{q}+\beta)(1-\delta)}|\varsigma|-\sum_{k=2}^{\infty}a_k|\varsigma|^k}. \tag{2.18}$$

Hence for $|\varsigma-\delta|<|\varsigma|<r$, (2.18) hold true if

$$\frac{\alpha(3-\nu)(1-\beta)c}{(\mathrm{q}+\beta)(1-\delta)}r^2+\sum_{k=2}^{\infty}(k+2-\nu)a_k r^{k+1} \leq \alpha(1-\nu),$$

and it follow that from (2.5), we may take

$$a_k\leq\frac{\alpha(1-\beta)(1-c)\lambda_k}{[k]_q^n(\mathrm{q}[k]_q+\beta)(1-\delta)},\qquad (k\geq 2),$$

where $\lambda_k \geq 0$ and $\sum_{k=2}^{\infty} \lambda_k \leq 1$.

For each fixed r, we choose the positive integer $k_0 = k_0(r)$ for which $\frac{\alpha(k_0+2-\nu)(1-\beta)(1-c)}{[k_0]_q^n(q[k_0]_q+\beta)(1-\delta)} r^{k_0+1}$, is maximal.

Then it follows that

$$\sum_{k=2}^{\infty} (k+2-\nu)a_k r^{k+1} \leq \frac{\alpha(k_0+2-\nu)(1-\beta)(1-c)}{[k_0]_q^n(q[k_0]_q+\beta)(1-\delta)} r^{k_0+1},$$

then F is starlike of order ν in $|\varsigma-\delta| < |\varsigma| < r_1$ provided that

$$\frac{\alpha(3-\nu)(1-\beta)c}{(q+\beta)(1-\delta)} r_1^2 + \frac{\alpha(k_0+2-\nu)(1-\beta)(1-c)}{[k_0]_q^n(q[k_0]_q+\beta)(1-\delta)} r_1^{k_0+1} \leq \alpha(1-\nu).$$

We find the value $r_1 = r_0(n,\alpha,\beta,c,\nu,k)$ and the corresponding integer $k_0(r_0)$ so that

$$\frac{\alpha(3-\nu)(1-\beta)c}{(q+\beta)(1-\delta)} r_0^2 + \frac{\alpha(k_0+2-\nu)(1-\beta)(1-c)}{[k_0]_q^n(q[k_0]_q+\beta)(1-\delta)} r_0^{k_0+1} = \alpha(1-\nu).$$

Then this value is the radius of starlikeness of order ν for function F belong to class $\Sigma_q^n[\delta,\alpha,\beta,c]$.

Theorem 7 *Let* $F(\varsigma) \in \Sigma_q^n[\delta,\alpha,\beta,c]$. *Then* $F(\varsigma)$ *is covex of order* ν $(0 \leq \nu < 1)$ *in* $|\varsigma-\delta| < |\varsigma| < r_2$, *where* r_2 *is the largest value for which*

$$\frac{\alpha(3-\nu)(1-\beta)c}{(q+\beta)(1-\delta)} r^2 + \frac{\alpha k_0(k_0+2-\nu)(1-\beta)(1-c)}{[k_0]_q^n(q[k_0]_q+\beta)(1-\delta)} r^{k+1} \leq \alpha(1-\nu), \tag{2.19}$$

for $k \geq 2$. The result is sharp for the function $F(\varsigma)$ given by (2.8).

Proof. By using the same technique in the proof of Theorem 6 we can show that

$$\left|\frac{(\varsigma-\delta)F''(\varsigma)}{F'(\varsigma)} + 2\right| \leq 1-\nu, \quad (|\varsigma| < r_2), \tag{2.20}$$

for $|\varsigma-\delta| < |\varsigma| < r_2$ with the aid of Theorem 2. Thus, we have the assertion of Theorem 7.

Theorem 8 *The class* $\Sigma_q^n[\delta,\alpha,\beta,c]$ *is closed under convex linear compination.*

Proof. Let $F(\varsigma)$ be defined by (2.4). Define the function $h(\varsigma)$ by

$$h(\varsigma) = \frac{\alpha}{\varsigma-\delta} + \frac{\alpha(1-\beta)c}{(q+\beta)(1-\delta)}\varsigma + \sum_{k=2}^{\infty} b_k\varsigma^k, \; b_k \geq 2. \tag{2.21}$$

Suppose that $F(\varsigma)$ and $h(\varsigma)$ are in the class $\Sigma_q^n[\delta,\alpha,\beta,c]$, we only need to prove that

$$G(\varsigma) = \zeta F(\varsigma) + (1-\zeta)h(\varsigma) \quad (0 \leq \zeta \leq 1), \tag{2.22}$$

also be in the class. Since

$$G(\varsigma) = \frac{\alpha}{\varsigma-\delta} + \frac{\alpha(1-\beta)c}{(q+\beta)(1-\delta)}\varsigma + \sum_{k=n+1}^{\infty} \{\zeta a_k + (1-\zeta)b_k\}\varsigma^k, \tag{2.23}$$

then

$$\sum_{k=2}^{\infty} [k]_q^n(q[k]_q+\beta)(1-\delta)\{\zeta a_k + (1-\zeta)b_k\} \leq \alpha(1-\beta)(1-c), \tag{2.24}$$

with the aid of Theorem 2. Hence $G(\varsigma) \in \Sigma_q^n[\delta,\alpha,\beta,c]$.

Theorem 9 *Let*

$$F_1(\varsigma) = \frac{\alpha}{\varsigma-\delta} + \frac{\alpha(1-\beta)c}{(q+\beta)(1-\delta)}\varsigma, \tag{2.25}$$

and

$$F_k(\varsigma) = \frac{\alpha}{\varsigma-\delta} + \frac{\alpha(1-\beta)c}{(q+\beta)(1-\delta)}\varsigma + \frac{\alpha(1-\beta)(1-c)}{[k]_q^n(q[k]_q+\beta)(1-\delta)}\varsigma^k, \tag{2.26}$$

for $k \geq 2$. Then $F(\varsigma) \in \Sigma_q^n[\delta,\alpha,\beta,c]$ iff

$$F(\varsigma) = \sum_{k=2}^{\infty} \eta_k F_k(\varsigma), \tag{2.27}$$

where $\eta_k \geq 0$ $(k \geq 2)$ and

$$\sum_{k=2}^{\infty} \eta_k \leq 1. \tag{2.28}$$

Proof. Let $F(\varsigma)$ be in the form (2.27). Then from (2.25), (2.26) and (2.28) we have

$$F(\varsigma)=\frac{\alpha}{\varsigma-\delta}+\frac{\alpha(1-\beta)c}{(\mathrm{q}+\beta)(1-\delta)}\varsigma$$

$$+\sum_{k=2}^{\infty}\frac{\alpha(1-\beta)(1-c)\eta_k}{[k]_q^n(\mathrm{q}[k]_q+\beta)(1-\delta)}\varsigma^k. \tag{2.29}$$

Since

$$\sum_{k=2}^{\infty}\frac{\alpha(1-\beta)(1-c)\eta_k}{[k]_q^n(\mathrm{q}[k]_q+\beta)(1-\delta)}.\frac{[k]_q^n(\mathrm{q}[k]_q+\beta)(1-\delta)}{\alpha(1-\beta)(1-c)}$$

$$=\sum_{k=2}^{\infty}\eta_k=1-\eta_1\leq 1, \tag{2.30}$$

then, from Theorem 2, $F(\varsigma)\in\Sigma_q^n[\delta,\alpha,\beta,c]$. Conversely, let $F(\varsigma)\in\Sigma_q^n[\delta,\alpha,\beta,c]$ and satisfies (2.9) for $k\geq 2$, then

$$\eta_k=\frac{[k]_q^n(\mathrm{q}[k]_q+\beta)(1-\delta)}{\alpha(1-\beta)(1-c)}a_k\leq 1, \tag{2.31}$$

and

$$\eta_1=1-\sum_{k=2}^{\infty}\eta_k. \tag{2.32}$$

This compelets the proof of the Theorem 9.

Corollary 3 *The extreme points of the class* $\Sigma_q^n[\delta,\alpha,\beta,c]$ *are the functions* $F_k(\varsigma)$ $(k\geq 2)$ *given by (2.25) and (2.26).*

Theorem 10 *If* $F(\varsigma)\in\Sigma_q^n[\delta,\alpha,\beta,c]$, *then*

$$\Sigma_q^n[\delta,\alpha,\beta,c]\subset N_{q,\xi}(F;q), \tag{2.33}$$

where the parameter ξ_q is given by

$$\xi_q=\frac{[2]_q\alpha(1-\beta)(1-c)}{[2]_q^n(\mathrm{q}[2]_q+\beta)(1-\delta)}. \tag{2.34}$$

Proof. For $F(\varsigma)\in\Sigma_q^n[\delta,\alpha,\beta,c]$, from (2.11) of Theorem 3 and in view of (1.10), we get (2.34).

3. ACKNOWLEDGEMENT

The authors thanks the referees of the paper for their valuable comments.

4. REFERENCES

1. M. K. Aouf, On a certain class of meromorphic univalent functions with positive coefficients, Rend. Mat. Appl., 7 (11) (1991), no. 2, 209-219.
2. M. K. Aouf and H. E. Darwish, Certain meromorphically starlike functions with positive and fixed second coefficients, Tr. J. of Mathematics., 21 (1997), 311- 316.
3. M. K. Aouf and S. M. Madian, Inclusion and properties neighbourhood for certain p-valent functions associated with complex order and q–p-valent Cătaş operator, J. Sci. Taibah Univer. Sci., 14(1) (2020) , 1226-1232.
4. M. K. Aouf, A. O. Mostafa and F. Y. AL-Quhali, Properties for class of β- uniformly univalent functions defined by Sălăgean type q-difference operator, Int. J. Open Probl. Complex Anal., 11 (2019), no. 2, 1–16.
5. J. Clunie, On meromorpic schlicht functions, J. Lond. Math.Soc., 34 (1959), 215-216.
6. B. A. Frasin and M. Darus, On certain meromorphic functions with positive coefficients, Southeast Asian Bull. Math., 28 (2004), 615-623.
7. F. Ghanim and M. Darus, On certain subclass of meromorphic univalent functions with fixed residue, Far East J. Math. Sci. (FJMS)., 26 (2007), 195- 207.
8. F. Ghanim and M. Darus, A new subclass of uniformly starlike and convex functions with negative coefficients, Int. J. Pure Appl. Math., 45 (4) (2008), 559-572.
9. F. Ghanim and M. Darus, On a certain subclass of meromorphic univalent functions with fixed second positive coefficients, Surveys in Mathematics and its Applications., 5 (2010), 49-60.
10. R. M. Goel and N. S. Sohi, On a class of meromorphic functions, Glas. Mat., 3. Ser., 17 (37) (1982), 19-28.
11. A.W. Goodman, Univalent functions and nonanalytic curves, Proc. Amer. Math. Soc., 8 (1957), 598-601.
12. Jinxi Ma, Extreme points and minimal outer area problem for meromorphic univalent functions, J. Math. Anal. Appl., 220 (2) (1998), 769-773.
13. J. Miller, Convex meromorphic mappings and related functions, Proc. Am. Math. Soc., 25 (1970), 220-228.
14. St. Ruscheweyh, Neighborhoods of univalent functions, Proc. Amer. Math. Soc., 81 (1981) 521-527.
15. H. M. Srivastava and S. Owa, Current Topics In Analytic Functions Theory, World Scientific, Singapore, 1992.
16. H. Tang, H. M. Zayed, A. O. Mostafa and M. K. Aouf, Fekete-Szegö problems for certain classes of meromorphic functions using q-derivative operator, J. Math. Resear. Appl., 38 (2018), no. 3, 236-246.
17. B. A. Uralgaddi and M. D. Ganigi, A certain class of meromorphically starlike functions with positive coefficients, Pure Appl. Math. Sci., 26 (1987), 75-81.

18. B. A. Uralegaddi and C. Somanatha, Certain differential operators for meromorphic functions, Houston J. Math., 17 (1991), 279-284.
19. B. A. Uralegaddi and C. Somanatha, New criteria for meromorphic starlike univalent functions, Bull. Austral. Math. Soc., 43 (1991), 137-140.