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Research Article

Unit Xgamma Distribution: Its Properties, Estimation and Application

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Abstract: A new one-parameter model for unit-interval datasets is introduced. The proposed distribution is termed "Unit Xgamma distribution." Some mathematical properties of the new distribution are derived. We also characterize it using truncated moments and a hazard function. Maximum likelihood, least-squares, weighted least-squares, Anderson-Darling, Cramer-von Mises, and maximum product spacing are among the five estimation methods used to estimate the parameter. A Monte Carlo simulation was used to test the efficacy of these developed estimators. The flexibility of the proposed distribution was assessed using water capacity data. The proposed unit Xgamma distribution can be used for bounded datasets as an alternative to the well-known competitive distributions available in the literature.

Keywords: Unit-Xgamma Moments Risk Measures Estimation Data Analysis.

1. INTRODUCTION

The bounded distributions, which are based on the unit interval, are useful for modeling the behavior of random variables with intervals (0,1). The Beta distribution [1] is considered a bounded distribution and is extensively used for modeling such data sets. There is always a need for other models for modeling bounded data sets. Some such important well-known distributions include Kumaraswamy distribution [2], Unit Burr-III [4], Unit Gompertz distribution [5], Unit Lindley distribution [6], Unit Gamma distribution [7], Unit Birnbaum-Saunders distribution [8], Unit-inverse Gaussian distribution [3], Unit Weibull distribution [9], Unit Logistic distribution [10], Unit modified Burr III distribution [11], unit Rayleigh distribution [12], Unit power-logarithmic distribution [13], odd Frechet power function distribution [14], Unit Burr XIII distribution [15], modified Kumaraswamy distribution [16], Unit Teissier distribution [17], inflated unit Birnbaum-Saunders distribution [18] and log-XLindley distribution because (i) it can be considered as an appropriate distribution to model the skewed data where other competent models available in the literature may not be adequately fitted; (ii) it can also be applied to model various real data sets in the fields of survival and industrial reliability.

The Xgamma distribution was introduced [20] to model lifetime data sets. The Xgamma distribution was derived using a finite gamma and exponential distribution mixture. A random variable Y has Xgamma distribution, if its probability density function (pdf) $\boldsymbol{g(x)}$ and cumulative distribution function (cdf) $\boldsymbol{G(x)}$ are, respectively, given by:

$$g(y;\theta)=\frac{\theta^2}{(1+\theta)}\left(1+\frac{\theta}{2}y^2\right)\exp(-\theta y),\quad y\geq 0. \tag{1}$$

$$G(y;\theta)=1-\left[\frac{1+\theta+\theta y+\frac{\theta^2y^2}{2}}{(1+\theta)}\right]\exp(-\theta y). \tag{2}$$

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where θ is the scale parameter.

The inspiration of this work is to propose a new distribution from the Xgamma distribution by a transformation $\boldsymbol{X = Y/(1+Y)}$, where Y has the Xgamma distribution.

- The proposed distribution comprises the behavior of the Beta distribution and provides better fits than some well-known lifetime probability distributions, such as the Kumaraswamy distribution.
- The proposed distribution is capable of data analysis of increasing hazard rate function.
- It can be viewed as the most suitable probability model for fitting negatively skewed data sets.

The paper is structured as follows: In Section 2, we present the Unit-Xgamma distribution (UXG) along with graphical representations of its pdf, cdf, and reliability function. In Section 3, some mathematical properties, including moments and associated measures, actuarial measures, and order statistics. The new distribution is also characterized based on truncated moments and hazard function. The parameter is estimated using five different estimation methods in Section 4. Monte-Carlo simulations are performed in Section 5 to investigate the performance of these estimators. The analysis of a real data set has been presented in Section 6. In the last section, we conclude our study.

2. UNIT-XGAMMA DISTRIBUTION

The Unit-Xgamma distribution is derived using the transformation $X = Y/(1+Y)$ with support on the unit interval. The cdf and pdf of the proposed distribution respectively are given by

$$F(x) = 1 - \frac{1+\theta+\theta\left(\frac{x}{1-x}\right)+\frac{\theta^2}{2}\left(\frac{x}{1-x}\right)^2}{(1+\theta)} e^{-\theta\left(\frac{x}{1-x}\right)}, \quad 0 < x < 1, \theta > 0. \tag{3}$$

and

$$f(x) = \left(\frac{\theta^2}{1+\theta}\right)\left[\frac{1}{(1-x)^2}\right]\left[1 + \frac{\theta}{2}\left(\frac{x}{1-x}\right)^2\right] e^{-\theta\left(\frac{x}{1-x}\right)}, \quad 0 < x < 1 \tag{4}$$

Figure 1 shows the shapes of the UXG distribution's pdf and cdf for various parameter values.

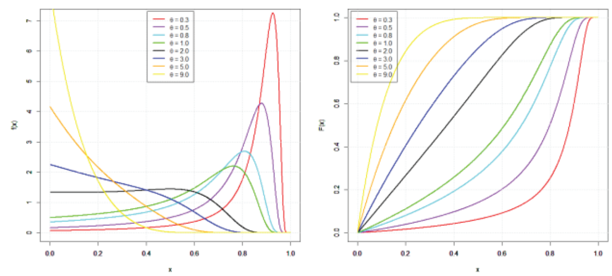

Fig. 1: Plots of pdf and cdf for UXG Distribution

$$S(x) = \frac{e^{\frac{-x\theta}{1-x}}\{2(1+\theta) - 2x(2+\theta) + x^2(2+\theta^2)\}}{2(1-x)^2(1+\theta)}, \tag{5}$$

$$h(x) = \frac{\theta^2\left\{1 + \frac{x^2\theta}{2(1-x)^2}\right\}}{(1-x)^2\left[1+\theta+\frac{x\theta}{1-x}+\frac{x^2\theta^2}{2(1-x)^2}\right]}. \tag{6}$$

Figure 2 shows the survival and hazard functions of the UXG distribution for various parameter values.

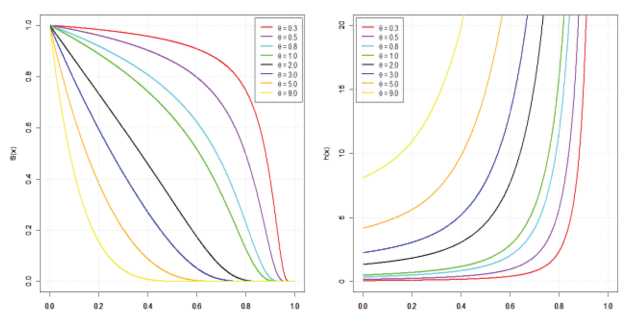

Fig. 2: Survival and hazard plots for UXG distribution

The shape of a pdf can be studied by limiting behavior at origin and one. The hazard function also showed the same results at the origin. That is

$$\lim_{x\to 0} h(x) = \lim_{x\to 0} f(x) = \frac{\theta^2}{1+\theta}$$

and shapes of pdf and hazard function at the upper limit are given by

$$\lim_{x\to\infty} f(x) = 0, \ \lim_{x\to\infty} h(x) = \infty$$

This shows that distribution begins from a point on the vertical axis. Now, it is needed to get more knowledge about its trend at origin by taking the limit of f'(x). f'(x) may be negative (positive), the pdf goes downward (upward) at the origin.

The first derivative of (4) is

$$\lim_{x\to 0}\frac{df(x)}{dx}=\frac{(2-\theta)\theta^2}{1+\theta}$$

For $\theta > 2$, the pdf goes down and upwards otherwise.

3. MATHEMATICAL PROPERTIES

3.1. Moments

If the random variable X is UXG distribution, then its r[th] moment about the origin can be given as

$$\mu_r' = \int_0^1 x^r\, f(x)\, dx$$

$$\mu_r' = \frac{\theta^2}{1+\theta}\int_0^1 \frac{x^r}{(1-x)^2} e^{-\theta\left(\frac{x}{1-x}\right)}dx + \frac{\theta^3}{2(1+\theta)}\int_0^1 \frac{x^{r+1}}{(1-x)^3} e^{-\theta\left(\frac{x}{1-x}\right)}dx \tag{7}$$

Considering first integral

$$I_1 = \left(\frac{\theta^2}{1+\theta}\right)\int_0^1 \frac{x^r}{(1-x)^2} e^{-\theta\left(\frac{x}{1-x}\right)}dx$$

Using transformation $z=\left(\frac{x}{1-x}\right)$, we get the expression

$$I_1 = \left(\frac{\theta^2}{1+\theta}\right)\int_0^\infty \left(\frac{z}{1+z}\right)^r e^{-\theta z}dz$$

Again taking $y = 1+z$

$$I_1 = \left(\frac{\theta^2 e^\theta}{1+\theta}\right)\int_1^\infty \left(1-\frac{1}{z}\right)^r e^{-\theta z}dz$$

Using expansion $(1-x)^p = \sum_{k=0}^\infty (-1)^k \binom{p}{k}(x)^k$

$$= \left(\frac{\theta^2 e^\theta}{1+\theta}\right)\sum_{k=0}^\infty (-1)^k\binom{r}{k}\int_1^\infty \left(\frac{1}{z}\right)^k e^{-\theta z}dz$$

$$= \left(\frac{\theta e^\theta}{1+\theta}\right)\sum_{k=0}^\infty (-1)^k\binom{r}{k}\theta^k\Gamma(1-k,\theta) \tag{8}$$

Now taking the second term of Eq. (7)

$$I_2 = \frac{\theta^3}{2(1+\theta)}\int_0^1 \frac{x^{r+2}}{(1-x)^4} e^{-\theta\left(\frac{x}{1-x}\right)}dx$$

Using transformation $z=\left(\frac{x}{1-x}\right)$, we get the following expression

$$I_2 = \frac{\theta^3 e^\theta}{2(1+\theta)}\int_1^\infty \left(1-\frac{1}{z}\right)^{r+2}\left(\frac{1}{z}\right)^2 e^{-\theta z}dz$$

$$= \frac{\theta^3 e^\theta}{2(1+\theta)}\sum_{k=0}^\infty (-1)^k\binom{r+2}{k}\int_1^\infty \left(\frac{1}{z}\right)^{k+2} e^{-\theta z}dz$$

$$= \frac{\theta^3 e^\theta}{2(1+\theta)}\sum_{k=0}^\infty (-1)^k\binom{r+2}{k}\theta^{k+1}\Gamma(-k-1,\theta) \tag{9}$$

The final expression of ordinary moments is

$$\mu_r' = \left(\frac{\theta e^\theta}{1+\theta}\right)\sum_{k=0}^\infty (-1)^k\binom{r}{k}\theta^k\Gamma(1-k,\theta) + \frac{\theta^3 e^\theta}{2(1+\theta)}\sum_{k=0}^\infty (-1)^k\binom{r+1}{k}\theta^k\Gamma(2-k,\theta). \tag{10}$$

The moment generating function of UXG distribution is obtained by using Eq. (10), given as

$$M_x(t) = E(e^{tX}) = \sum_{r=0}^\infty \frac{t^r}{r!}E(X^r). \tag{11}$$

The numerical values of mean, variance, dispersion index, skewness, and kurtosis for some selected values of parameters are presented in Table 1. The behavior of UXG distribution is negatively for lower values of $\theta \le 1$ and positively skewed for $\theta > 1$. The UXG distribution is leptokurtic for $\theta \le 0.5\ or\ \theta > 5$ and $\theta = 5$ model is mesokurtic. Also, for $0.5 < \theta < 5$, UXG model is platykurtic.

3.2. Actuarial Measures

In this subsection we derived two risk measures, value at risk (VaR) and tail value at risk (TVaR). For more information about actuarial measures, readers can consult the following studies [21] [22] [23].

The VaR of UXG is derived as $x_p = F^{-1}(t)$, where t is the solution of the equation

Table 1: Some descriptive measures for some specific values of θ.

θ	μ	σ^2	DI	Skewness	Kurtosis
0.3	0.82190	0.02792	0.03397	-2.40551	9.18987
0.5	0.72439	0.04266	0.05889	-1.53032	4.76441
0.8	0.61147	0.05271	0.08620	-0.90568	2.85768
1.0	0.55274	0.05452	0.09864	-0.64885	2.36971
2.0	0.36979	0.04554	0.12315	0.04662	1.91532
3.0	0.27578	0.03378	0.12249	0.40451	2.19210
4.0	0.21908	0.02527	0.11535	0.64096	2.57963
5.0	0.18134	0.01939	0.10693	0.81396	2.97398
7.0	0.13445	0.01228	0.09134	1.05525	3.69344
9.0	0.10660	0.00840	0.07880	1.21769	4.29919
12	0.08117	0.00526	0.06480	1.38233	5.02381
15	0.06547	0.00359	0.05483	1.49321	5.58091
20	0.04945	0.00215	0.04348	1.61401	6.25796

Table 2: The risk measures (VaR and TVaR) for the UXG distribution

θ	Significance level	VaR_p	$TVaR_p$
0.50	0.70	0.85683	0.26838
	0.75	0.86887	0.22525
	0.80	0.88061	0.18149
	0.85	0.89244	0.13719
	0.90	0.90506	0.09224
	0.95	0.92016	0.04664
	0.99	0.94051	0.00947
1.0	0.70	0.72049	0.23814
	0.75	0.74382	0.20153
	0.80	0.76660	0.16378
	0.85	0.78961	0.12487
	0.90	0.81414	0.08479
	0.95	0.84349	0.04339
	0.99	0.88306	0.00896
2.0	0.70	0.51093	0.18823
	0.75	0.54572	0.16181
	0.80	0.58118	0.13364
	0.85	0.61831	0.10366
	0.90	0.65914	0.07175
	0.95	0.70943	0.03760
	0.99	0.77929	0.00804
5.0	0.70	0.24178	0.10797
	0.75	0.27041	0.09519
	0.80	0.30284	0.08087
	0.85	0.34069	0.06482
	0.90	0.38733	0.04666
	0.95	0.45238	0.02579
	0.99	0.55669	0.00598

$$\left(\frac{x}{1-x}\right)+\frac{\theta}{2}\left(\frac{x}{1-x}\right)^{2}e^{-\theta\left(\frac{x}{1-x}\right)}=-\frac{p(1+\theta)}{\theta}$$

TVaR of X is defined as

$$TVaR_{p}(x)=\frac{1}{1-p}\int_{VaR}^{\infty}xf(x)\,dx \tag{12}$$

Using Eq. (4) in (12), we get

$$TVaR_{p}(x)=\frac{e^{\theta}\left(\frac{\theta^{2}}{1+\theta}\right)}{1-p}\left[\frac{e^{\left(\frac{1}{1-VaR}\right)}}{\theta}-\Gamma\left(0,\frac{1}{1-VaR}\right)+\frac{\theta}{2}\sum_{k=0}^{\infty}(-1)^{k}\binom{3}{k}\theta^{k+1}\Gamma\left(-1-k,\frac{1}{1-VaR}\right)\right]. \tag{13}$$

Some numerical values of VaR and TVaR are presented in Table 2.

3.3. Order Statistics

Order statistics is commonly used and performed in the statistical literature. Let $X_1, X_2, \dots, X_n$ represent r.v. with cdf $F(x)$. If $X_{1:n}, X_{2:n}, \dots, X_{n:n}$ are the related ordered random samples of size *n*, then the pdf of r^{th} order statistic is given as

$$f_{r:n}=W[F(x)]^{r-1}[1-F(x)]^{n-r}f(x)$$

where $W=\frac{n!}{(r-1)!\,(n-r)!}$

By using binomial expansion

$$=W\sum_{l=0}^{n-r}(-1)^{l}\binom{n-r}{l}[F(x)]^{r+l-1}f(x)$$

$$=W\sum_{l=0}^{n-r}(-1)^{l}\binom{n-r}{l}\times$$

$$\left[1-\frac{1+\theta+\theta\left(\frac{x}{1-x}\right)+\frac{\theta}{2}\left(\frac{x}{(1-x)^{2}}\right)^{2}}{1+\theta}e^{-\theta\left(\frac{x}{1-x}\right)}\right]^{r+l-1}\times$$

$$\left(\frac{\theta^{2}}{1+\theta}\right)\left[\frac{1}{(1-x)^{2}}\right]\left[1+\frac{\theta}{2}\left(\frac{x}{(1-x)^{2}}\right)^{2}\right]e^{-\theta\left(\frac{x}{1-x}\right)}$$

$$=W\frac{\theta^{2}}{1+\theta}\sum_{l=0}^{n-r}(-1)^{l}\binom{n-r}{l}\frac{1}{(1-x)^{2}}\times\left[1+\frac{\theta}{2}\left(\frac{x}{(1-x)^{2}}\right)^{2}\right]e^{-\theta\left(\frac{x}{1-x}\right)}$$

$$\times\sum_{s=0}^{r+l+1}(-1)^{s}\binom{r+l+1}{s}\left[\frac{1+\theta+\theta\left(\frac{x}{1-x}\right)+\frac{\theta}{2}\left(\frac{x}{(1-x)^{2}}\right)^{2}}{1+\theta}\right]^{s}e^{-\theta s\left(\frac{x}{1-x}\right)}$$

$$=W\frac{\theta^{2}}{1+\theta}\sum_{l=0}^{n-r}\sum_{s=0}^{r+l+1}(-1)^{l+s}\binom{n-r}{l}\binom{r+l+1}{s}\frac{1}{(1-x)^{2}}\left[1+\frac{\theta}{2}\left(\frac{x}{(1-x)^{2}}\right)^{2}\right]\left[\frac{1+\theta+\theta\left(\frac{x}{1-x}\right)+\frac{\theta}{2}\left(\frac{x}{(1-x)^{2}}\right)^{2}}{1+\theta}\right]^{s}e^{-\theta(s+1)\left(\frac{x}{1-x}\right)}$$

By using exponential expansion

$$f_{r:n}=W\frac{\theta^{2}}{1+\theta}\sum_{l=0}^{n-r}\sum_{s=0}^{r+l+1}\sum_{q=0}^{\infty}(-1)^{l+s+q}\frac{\left(\theta(s+1)\right)^{q}}{q!}\binom{n-r}{l}\binom{r+l+1}{s}\times\frac{x^{q}}{(1-x)^{q+2}}\times\left[1+\frac{\theta}{2}\left(\frac{x}{(1-x)^{2}}\right)^{2}\right]\left[\frac{1+\theta+\theta\left(\frac{x}{1-x}\right)+\frac{\theta}{2}\left(\frac{x}{(1-x)^{2}}\right)^{2}}{1+\theta}\right]^{s}.$$

3.4. Characterizations

The presence of a new stochastic function must be validated against the underlying model's requirements. Glänzel [24] suggests that studying characterizations could be useful in this approach. The ratio of two truncated moments is used to characterize the UXG distribution. So we use the idea of Glänzel to characterize the UXG distribution.

Proposition 1: Let X: $\Omega \rightarrow (0,1)$ be distributed as Eq. (4) and

$$q_{1}(x)=\left\{1+\frac{x^{2}\theta}{2(1-x)^{2}}\right\}^{-1}$$

$$q_{2}(x)=q_{1}(x)\ \ e^{\frac{x\theta}{x-1}},\quad x>0. \tag{18}$$

The random variable X follows UXG if and only if the function η defined in Theorem [24] is of the form

$$\eta(x)=\frac{1}{2}e^{\frac{x\theta}{x-1}} \tag{19}$$

Proof:

$$\left(1-F(x)\right)E[q_1(X)|X\geq x]=\int_x^1 \frac{e^{\frac{x\theta}{x-1}}\theta^2}{(x-1)^2(1+\theta)}dx$$

$$=\frac{e^{\frac{x\theta}{x-1}}\theta}{(1+\theta)},0<x<1$$

Similarly,

$$\left(1-F(x)\right)E[q_2(X)|X\geq x]=\frac{e^{\frac{2x\theta}{x-1}}\theta}{2(1+\theta)},\ \ 0<x<1$$

Now

$$\eta(x)=\frac{E[q_2(X)|X\geq x]}{E[q_1(X)|X\geq x]}=\frac{1}{2}e^{\frac{x\theta}{x-1}}$$

Completes the proof.

and $\eta(x)q_1(x)-q_2(x)=q_1(x)\left[\eta(x)-e^{\frac{x\theta}{x-1}}\right]$

$$=\frac{e^{\frac{x\theta}{-1+x}}(-1+x)^2}{2+x(-4+x(2+\theta))}<0$$

$$\acute{s}(x)=\frac{\acute{\eta}(x)q_1(x)}{\eta(x)q_1(x)-q_2(x)}=\frac{\theta}{(x-1)^2}$$

and hence

$$s(x)=-\ln\left(e^{-\frac{x\theta}{1-x}}\right),\quad 0<x<1 \qquad (20)$$

Now by Proposition (1), X has density (4).

Corollary 1: Assume $X:\Omega\to(0,1)$ be a continuous random variable, then $q_1(x)$ is the same as in Proposition 1. The pdf of X is (4) if and only if the differential equation is satisfied by functions $q_2(x)$ and η(x) stated in Theorem [24].

$$\frac{\acute{\eta}(x)q_1(x)}{\eta(x)q_1(x)-q_2(x)}=\frac{\theta(2+x(-4+x(2+\theta)))}{2(-1+x)^4},\quad 0<x<1$$

(21) The general solution of the differential Eq. (21) is

$$\eta(x)=\left[e^{\frac{x\theta}{-1+x}}\right]^{-1}$$

$$\times\left[\int_0^x \frac{-\theta}{(1-x)^2}[q_2(x)\{q_1(x)\}^{-1}]dx+D\right],$$

D is the constant in this equation. Note that in Proposition 1 with $D=0$, $q_1(x),\ q_2(x)\ and\ \eta(x)$ are set of functions that satisfy the differential Eq.(21).

The hazard function-related characterization of the UXG distribution is now given. The hazard function h(x) is known to satisfy the following differential equation.

$$\frac{\acute{f}(x)}{f(x)}=\frac{\acute{h}(x)}{h(x)}-h(x)$$

Proposition 2: If $h(x)$ satisfies the differential equation, then X is a continuous random variable range between 0 and 1 with pdf Eq. (4).

$$\acute{h}(x)-2(1-x)^{-1}h(x)$$
$$=\frac{2\left(-2+x\left(6+x(-4+\theta)\right)\right)\theta^3}{(x-1)^2\left(2(1+\theta)-2x(2+\theta)+x^2(2+\theta^2)\right)^2}. \qquad (22)$$

under the boundary conditions h(0)≥0.

Proof: If r.v. X has the HR given in (6) then

$$\acute{h}(x)$$
$$=\frac{1}{(x-1)^3(2(1+\theta)-2x(2+\theta)+x^2(2+\theta^2))^2)}$$
$$\times\begin{bmatrix}-((2\theta^2(4(x-1)^4+2(x-1)^2(1+x^2)\theta\\ +(x-1)x^2(2x-5)\theta^2+x^4\theta^3))\end{bmatrix}$$

and substitution of the above result in

$\acute{h}(x)-2(1-x)^{-1}h(x)$ and the result follows.

Conversely, if Eq. (22) holds then

$$\frac{d}{dx}[(1-x)^2h(x)]$$
$$=\frac{d}{dx}\left(\frac{\theta^2(1+\frac{x^2\theta}{2(1-x)^2})}{1+\theta+\frac{x\theta}{1-x}+\frac{x^2\theta^2}{2(1-x)^2}}\right)$$

$$h(x)=\frac{\theta^2(1+\frac{x^2\theta}{2(1-x)^2})}{(1-x)^2(1+\theta+\frac{x\theta}{1-x}+\frac{x^2\theta^2}{2(1-x)^2})}+C$$

which implies C=0.

4. PARAMETER ESTIMATION

In this section, we investigate the estimation of unknown parameter θ of UXG distribution using different well-known estimation methods. The considered estimation methods are; maximum likelihood, Anderson Darling, Cramer Von Mises, ordinary least squares, weighted least square, and maximum product spacing. For more detail reader can consult following studies [25], [26] [27]. Now onwards, a random sample is denoted by $x_1, x_2, x_3, \ldots, x_n$ from the $UXG(\theta)$ distribution of size n. The nonlinear equations of different estimation methods can be solved using statistical software (e.g., R, Mathematica). We used R in this study.

4.1. Maximum Likelihood Estimation

The most favorable parametric estimating method is the MLE method. The reason is described by theoretical acceptance of the estimators' limiting characteristics, such as consistency, efficiency, and asymptotic normality. The method of MLE of the UXG model is given below. The MLE of θ is obtained by maximizing the log-likelihood function for θ, the log-likelihood function $l(\theta)$ is equal to

$$l(\theta) = n\log\left(\frac{\theta^2}{1+\theta}\right) - 2\sum_{i=1}^{n}\log[1-x_i] + \sum_{i=1}^{n}\log\left[1+\frac{\theta}{2}\left(\frac{x_i}{1-x_i}\right)^2\right] + \theta\sum_{i=1}^{n}\left(\frac{x_i}{1-x_i}\right).$$

$U_\theta = \partial l(\theta)/\partial\theta$ is given below

$$U_\theta = \frac{n(1+\theta)\left[\frac{2\theta}{1+\theta} - \frac{\theta^2}{(1+\theta)^2}\right]}{\theta^2} + \sum_{i=1}^{n}\frac{x_i^2}{2(1-x_i)^2\left(1+\frac{\theta x_i^2}{2(1-x_i)^2}\right)} - \sum_{i=1}^{n}\left(\frac{x_i}{1-x_i}\right)$$

equating U_θ to zero and it can be solved using statistical software R.

4.2. Least Squares and Weighted Least Squares

We now take into account the methods of ordinary least squares (OLS) estimation and weighted least-squares (WLS) estimation. OLS estimation method was firstly presented by Swain (1988). It is a nonlinear method of estimation, especially when the MLEs cannot be obtained in an explicit form. The OLS of θ can be obtained by minimizing the least square function L(θ)

$$L(\theta) = \sum_{i=1}^{n}\left(F\left(x_{(i)};\theta\right) - \frac{i}{n+1}\right)^2.$$

with respect to θ, where x(i), (i=1,2,3,…..,n) is the ith element of the ordered observations $x_1, x_2, x_3 \ldots, x_n$ and $\hat{F}(.)$ is empirical CDF of i[th] observation. i.e., $\hat{F}(.) = \frac{i}{n+1}$.
By using Eq. (3) and $\hat{F}(.)$, we have

$$= \sum_{i=1}^{n}\left(1 - \frac{1+\theta+\theta\left(\frac{x_{(i)}}{1-x_{(i)}}\right)+\frac{\theta^2}{2}\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^2}{(1+\theta)} e^{-\theta\left(\frac{x_{(i)}}{1-x_{(i)}}\right)} - \frac{i}{n+1}\right)^2.$$

Thus, the LSE can be obtained by equating the equation to zero, i.e., $\partial L(\theta)/\partial\theta = 0$

$$\frac{\partial L(\theta)}{\partial\theta} = 2\sum_{i=1}^{n}\eta_i\left(1 - \frac{i}{1+n} - \frac{e^{-\frac{\theta x_{(i)}}{1-x_{(i)}}}\left(1+\theta+\frac{\theta x_{(i)}}{1-x_{(i)}}+\frac{\theta^2 x_{(i)}^2}{2(1-x_{(i)})^2}\right)}{1+\theta}\right).$$

where

$$\eta_i = \frac{-1}{2(x_{(i)}-1)^3(1+\theta)^2} \times \left(x_i \theta e^{\frac{x_{(i)}\theta}{x_{(i)}-1}} \begin{Bmatrix} 2(2+\theta) - x_{(i)} \\ (8+3\theta) + x_{(i)}{}^2(4+2\theta+\theta^2) \end{Bmatrix} \right) \quad (23)$$

The weighted least square estimate (WLS) of θ, can be obtained by minimizing the weighted least square function with respect to θ, defined by

$$WLS(\theta) = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_{(i)};\theta) - \frac{i}{n+1}\right]^2$$

$$= \sum_{i=1}^{n} \frac{(n+1)^2(n+2)^2}{i(n-i+1)} \times \left[1 - \frac{1+\theta+\theta\left(\frac{x_{(i)}}{1-x_{(i)}}\right) + \frac{\theta^2}{2}\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^2}{(1+\theta)} e^{-\theta\left(\frac{x_{(i)}}{1-x_{(i)}}\right)} - \frac{i}{n+1}\right]$$

$\partial WLS(\theta)/_{\partial\theta} = 0$ gives the WLSQE of θ.

$$\frac{\partial WLS(\theta)}{\partial\theta} = 2\sum_{i=1}^{n} \eta_i \frac{(n+1)^2(n+2)}{i(n-i+1)} \times \left(1 - \frac{i}{1+n} - \frac{e^{-\frac{\theta x_{(i)}}{1-x_{(i)}}}\left(1+\theta+\frac{\theta x_{(i)}}{1-x_{(i)}} + \frac{\theta^2 x_{(i)}^2}{2(1-x_{(i)})^2}\right)}{1+\theta}\right)$$

where η_i is given by Eq. (23).

4.3. Cramer-von Misses Estimation

Another method for obtaining estimates is the Cramer-von Mises (CVM) estimates by minimizing its function with respect to θ. It is defined by

$$C(\theta) = \frac{1}{12n} + \sum_{i=1}^{n} \left[F(x_{(i)};\theta) - \frac{2i-1}{2n}\right]^2$$

$$= \sum_{i=1}^{n} \left[1 - \frac{1+\theta+\theta\left(\frac{x_{(i)}}{1-x_{(i)}}\right) + \frac{\theta^2}{2}\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^2}{(1+\theta)} e^{-\theta\left(\frac{x_{(i)}}{1-x_{(i)}}\right)} - \frac{2i-1}{2n}\right]^2 + \frac{1}{12n}$$

Thus, the CVM estimates are obtained by the given equation: $\partial C(\theta)/\partial\theta = 0$, where

$$\frac{\partial C(\theta)}{\partial\theta} = 2\sum_{i=1}^{n} \eta_i \left(1 - \frac{1+\theta+\theta\left(\frac{x_{(i)}}{1-x_{(i)}}\right) + \frac{\theta^2}{2}\left(\frac{x_{(i)}}{1-x_{(i)}}\right)^2}{(1+\theta)} e^{-\theta\left(\frac{x_{(i)}}{1-x_{(i)}}\right)} - \frac{2i-1}{2n}\right)$$

where η_i is given by Eq. (23).

The CVE shows that this estimator's bias (θ) is lower than those of other minimum distance estimators.

4.4. Anderson-Darling Estimation

The Anderson-Darling (AD) estimate of θ can be obtained by minimizing the following function, with respect to θ, which is given by

$$A(\theta) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1) \times \left[\log F(x_{(i)};\theta) + \log[1 - F(x_{(n+1-i)};\theta)]\right]$$

Thus, the AD estimate can be determined by solving the following equation: $\partial A(\theta)/\partial\theta = 0$,

$$\frac{\partial A(\theta)}{\partial\theta} = -\frac{1}{n}\sum_{i=1}^{n}(2i-1) \times \left(\frac{\eta_i}{F(x_{(i)};\theta)} - \frac{\eta_{n+1-i}}{1-F(x_{(n+1-i)};\theta)}\right).$$

where η_i is given in Eq. (23).

Note that all estimation methods can be obtained by using numerical methods.

4.5. Maximum Product Spacing Estimation

For $u = 1,2,3,\dots,h+1$, assume $D_u(\theta) = F(x_{(u)} \mid \theta) - F(x_{(u-1)} \mid \theta)$, be the uniform spacings of a random sample from the UXG model, where $F(x_{(0)}|\theta) = 0$, $F(x_{(h+1)}|\theta) = 1$ and $\sum_{r=1}^{h+1} D_u(\theta) = 1$. The MPSE of the parameter θ, say $\hat{\lambda}$, can be estimated by maximizing the geometric mean of the spacings

$$\mathrm{MPSE}(\lambda) = \left[\prod_{u=1}^{h+1} D_u(\lambda)\right]^{\frac{1}{h+1}}$$

with respect to the parameter θ.

5. SIMULATION STUDY

In this section, some simulation studies are presented for the comparison of different estimation methods such as maximum likelihood estimation (MLE), least-squares estimation (OLS), weighted least square estimation (WLS), Cramer-von Mises estimation (CVM), Anderson-Darling estimation (AD) and maximum product spacing (MPS). In simulation studies, both small and large sample sizes are considered. We evaluate the performance of the estimators by bias, mean squared errors.

We consider the UXG model, and data are simulated using $N = 10{,}000$ with sample sizes n=20, 50, 80, 100, and 200 for some specific values of a parameter. Simulated bias and MSE are given in Tables 2-8.

Table 2: The simulated biases and MSEs of the UXG model for (θ=0.3)

n		ML	AD	CVM	OLS	WLS	MPS
20	Bias	0.00685	0.00467	0.00646	0.00508	0.00454	0.00499
	MSE	0.00185	0.00201	0.00231	0.00228	0.00214	0.00168
50	Bias	0.00259	0.00184	0.00259	0.00204	0.00189	0.00341
	MSE	0.00074	0.00081	0.00090	0.00089	0.00083	0.00071
80	Bias	0.00057	0.00019	0.00018	0.00017	0.00008	0.00354
	MSE	0.00044	0.00048	0.00052	0.00052	0.00049	0.00044
100	Bias	0.00191	0.00111	0.00137	0.00109	0.00120	0.00156
	MSE	0.00034	0.00038	0.00042	0.00042	0.00039	0.00033
200	Bias	0.00031	0.00027	0.00058	0.00044	0.00033	0.00167
	MSE	0.00017	0.00019	0.00021	0.00020	0.00019	0.00017

Table 3: The simulated biases and MSEs of the UXG model for (θ=0.5)

n		ML	AD	CVM	OLS	WLS	MPS
20	Bias	0.01231	0.00812	0.01221	0.00662	0.00618	0.00959
	MSE	0.00597	0.00662	0.00766	0.00737	0.00676	0.00534
50	Bias	0.00477	0.00264	0.00368	0.00269	0.00402	0.00658
	MSE	0.00217	0.00245	0.00274	0.00271	0.00246	0.00207
80	Bias	0.00260	0.00169	0.00260	0.00185	0.00164	0.00479
	MSE	0.00133	0.00150	0.00164	0.00163	0.00151	0.00128
100	Bias	0.00187	0.00155	0.00210	0.00144	0.00172	0.00448
	MSE	0.00106	0.00117	0.00131	0.00129	0.00120	0.00103
200	Bias	0.00122	0.00095	0.00091	0.00097	0.00093	0.00248
	MSE	0.00051	0.00059	0.00064	0.00063	0.00061	0.00052

Table 4: The simulated biases and MSEs of the UXG model for (θ=1.0)

n		ML	AD	CVM	OLS	WLS	MPS
20	Bias	0.03171	0.01993	0.02883	0.01879	0.01888	0.01950
	MSE	0.02842	0.03072	0.03716	0.03601	0.03413	0.02487
50	Bias	0.01168	0.00620	0.01049	0.00872	0.00720	0.01592
	MSE	0.00994	0.01135	0.01291	0.01339	0.01188	0.00970
80	Bias	0.00728	0.00368	0.00688	0.00514	0.00465	0.01012
	MSE	0.00632	0.00711	0.00806	0.00815	0.00733	0.00609
100	Bias	0.00595	0.00471	0.00545	0.00405	0.00392	0.00901
	MSE	0.00499	0.00564	0.00629	0.00628	0.00578	0.00474
200	Bias	0.00315	0.00121	0.00260	0.00178	0.00121	0.00518
	MSE	0.00239	0.00281	0.00311	0.00310	0.00281	0.00240

Table 5: The simulated biases and MSEs of the UXG model for (θ=1.5)

n		ML	AD	CVM	OLS	WLS	MPS
20	Bias	0.04783	0.03207	0.04273	0.03108	0.02652	0.02727
	MSE	0.07092	0.07885	0.09671	0.09256	0.08777	0.06315
50	Bias	0.01895	0.01320	0.01862	0.01269	0.01408	0.02325
	MSE	0.02547	0.02924	0.03307	0.03326	0.03040	0.02420
80	Bias	0.01113	0.00968	0.01201	0.01085	0.00763	0.01505
	MSE	0.01526	0.01808	0.02066	0.02026	0.01837	0.01502
100	Bias	0.00993	0.00671	0.00696	0.00737	0.00393	0.01496
	MSE	0.01205	0.01414	0.01619	0.01611	0.01413	0.01224
200	Bias	0.00378	0.00233	0.00380	0.00130	0.00371	0.00864
	MSE	0.00597	0.00697	0.00770	0.00768	0.00721	0.00600

Table 6: The simulated biases and MSEs of the UXG model for (θ=2.0)

n		ML	AD	CVM	OLS	WLS	MPS
20	Bias	0.07189	0.04638	0.07121	0.05729	0.05001	0.04358
	MSE	0.15423	0.17641	0.21967	0.21614	0.19800	0.13118
50	Bias	0.02671	0.01201	0.01794	0.01226	0.01291	0.03152
	MSE	0.04729	0.05204	0.05935	0.05893	0.05326	0.04445
80	Bias	0.02303	0.01623	0.02137	0.01776	0.01737	0.01769
	MSE	0.03029	0.03439	0.03949	0.03924	0.03534	0.02860
100	Bias	0.00860	0.00345	0.00749	0.00461	0.00434	0.02521
	MSE	0.02393	0.02754	0.03091	0.03080	0.02803	0.02352
200	Bias	0.00634	0.00722	0.01055	0.00910	0.00772	0.01312
	6MSE	0.01200	0.01412	0.01608	0.01603	0.01424	0.01186

Table 7: The simulated biases and MSEs of the UXG model for (θ=3.0)

n		ML	AD	CVM	OLS	WLS	MPS
20	Bias	0.12381	0.07841	0.04273	0.08121	0.08188	0.07461
	MSE	0.36624	0.39167	0.09671	0.47460	0.45088	0.30881
50	Bias	0.04934	0.03282	0.01862	0.03397	0.03251	0.04010
	MSE	0.12587	0.14534	0.03307	0.16815	0.15049	0.11975
80	Bias	0.02901	0.01785	0.01201	0.01465	0.01972	0.03396
	MSE	0.07490	0.08930	0.02066	0.09717	0.09358	0.07316
100	Bias	0.02240	0.01198	0.00696	0.01459	0.01432	0.03357
	MSE	0.05912	0.06956	0.01619	0.07882	0.07222	0.05711
200	Bias	0.02910	0.03448	0.00770	0.03867	0.03437	0.02904
	6MSE	0.01201	0.00852	0.00380	0.00968	0.00636	0.02065

Table 8: The simulated biases and MSEs of the UXG model for (θ=5.0)

n		ML	AD	CVM	OLS	WLS	MPS
20	Bias	0.21736	0.13968	0.20750	0.16371	0.13319	0.11997
	MSE	1.14258	1.28215	1.62899	1.57041	1.41972	0.99647
50	Bias	0.08943	0.06451	0.06921	0.06478	0.04772	0.08697
	MSE	0.39571	0.46983	0.52545	0.53176	0.47153	0.37371
80	Bias	0.05461	0.03895	0.04216	0.03479	0.03175	0.07200
	MSE	0.24553	0.28065	0.31496	0.31822	0.28564	0.23001
100	Bias	0.04117	0.02626	0.04053	0.03854	0.02625	0.05860
	MSE	0.19236	0.22304	0.26136	0.25799	0.22927	0.18659
200	Bias	0.09231	0.10894	0.12467	0.12419	0.10977	0.09221
	MSE	0.01840	0.01645	0.02161	0.01445	0.01007	0.02868

The following observations can be made from Table 2-8.

1. The estimators of θ show the characteristic of consistency i.e., the MSE decreases as the sample size (n) increases.
2. The bias of $\hat{\theta}$ drops with increasing n for all the methods of estimation.
3. The bias of $\hat{\theta}$ generally increases with increasing θ for any given θ and n.

6. MODELING REAL DATA

In this section, we present an analysis on a real data set to display the modeling behaviour of UXG distribution in comparison with the competitive distributions. The data set comprises water capacity month-wise from the Shasta reservoir in California in the month of February from 1991–2010. The observations are: 0.338936, 0.431915, 0.759932, 0.724626, 0.757583, 0.811556, 0.785339, 0.783660, 0.815627, 0.847413, 0.768007, 0.843485, 0.787408, 0.849868, 0.695970, 0.842316, 0.828689, 0.580194, 0.430681 and 0.742563.

To begin, we plot the total time test (TTT) plot and the box plot in Figure 3 to analyze the underlying distribution of the given data set. The TTT plot depicts the empirical hazard rate function as it increases. The data is negatively skewed, as seen by the box plot. To see if this data set follows the UXG and most popular lifespan distributions, we use the Anderson-Darling (AD), Cramer von-Misses (CVM) and Kolmogorov-Smirnov (KS) test statistics.

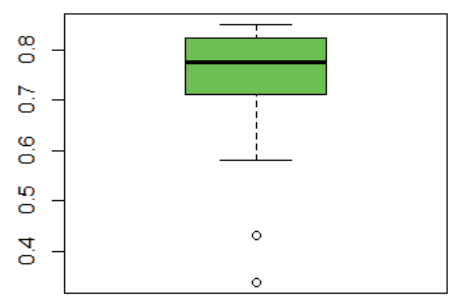

Fig. 3. Box plot for water capacity data

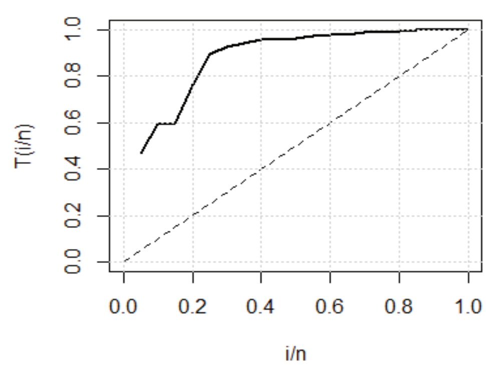

Fig. 4. TTT plot for water capacity data

Now, we apply UXG, Kumaraswamy, Beta Distribution, Unit Burr III, Unit Lindley, Topp Leon distributions as a model to the water capacity data set and Table 10 provides the ML estimates

Table 9: Some competitive models for the UXG distribution

Sr.	Distribution	Author(s)	References
1	Beta distribution (BD)	Gupta & NADarajah, (2004)	[1]
2	Kumaraswamy distribution (KwD)	Kumaraswamy, (1980)	[2]
3	Unit Burr III distribution (UBIII)	Modi & Gill, (2019)	[3]
4	Topp leon distribution (TLD)	Nadarajah & Kotz, (2003)	[24]
5	Unit Lindley distribution (ULin)	Mazucheli et al., (2019)	[5]

Table 10: ML estimates and Model selection measures

Model	MLEs	LogL	AIC	BIC	CVM	AD	KS
KwD(α,β)	6.3475 (1.5574) 4.4890 (2.0406)	13.475	-22.949	-20.958	0.2505	1.5325	0.2209
BD(α,β)	7.3157 (2.3181) 2.9099 (0.8755)	12.562	-21.124	-19.132	0.2861	1.6137	0.2359
UBIIID(α,β)	3.4965 (0.8154) 1.5291 (0.2243)	11.077	-18.154	-16.163	0.2999	1.6658	0.2379
TLD(λ)	8.6653 (1.9376)	11.588	-21.175	-20.179	0.3319	1.7858	0.2549
ULinD(θ)	0.4958 (0.0806)	13.827	-25.654	-24.659	6.0498	1.8333	0.7484
UXGD(θ)	**0.6810 (0.0985)**	**14.024**	**-26.048**	**-25.052**	**0.2452**	**1.4224**	**0.2066**

According to Table 10, the UXG distribution fits the water capacity data set better than the other competitive distributions since the accuracy measures for determining the ideal distribution for a given data set are smaller. The histogram of the water capacity data set and the fitted densities, estimated HR and PP plots are displayed in Figure 4. From Figure 4, it is observed that the fitted density of UXG distribution fits the data well and fitted (estimated) HR shows an increasing trend which is also confirmed by the TTT plot.

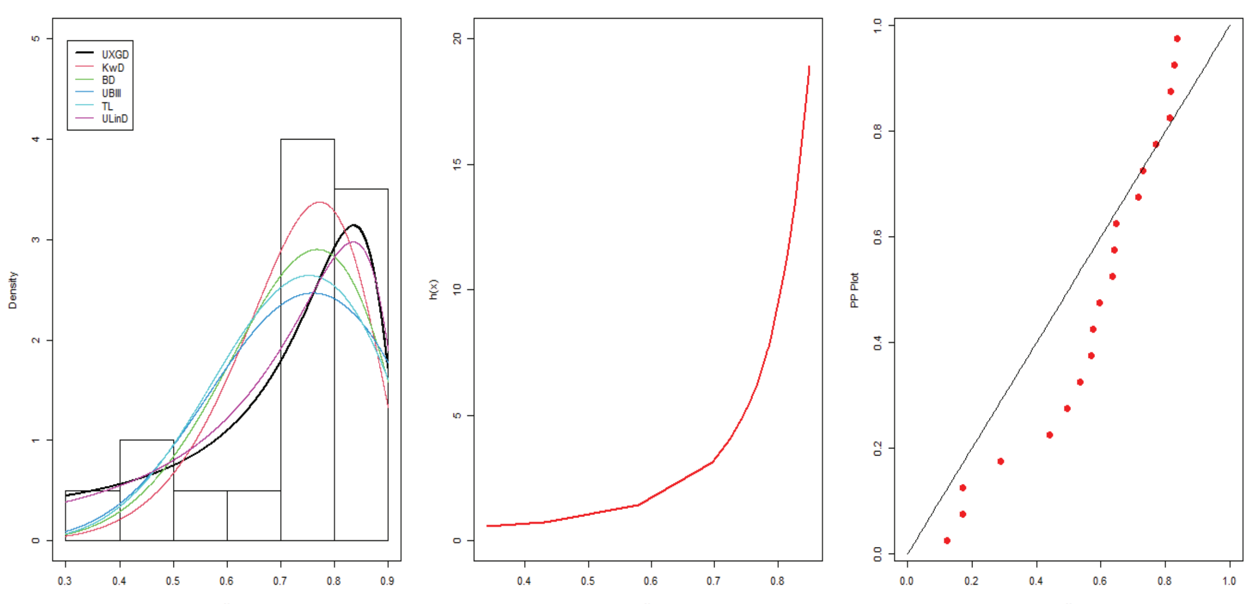

Fig. 4: (Left) Fitted densities, (middle) estimated HRF and (right) PP plot for the data set.

Figure 5 gives the value of the MLE of parameter θ. The maximum value of the LogL function is at θ = 0.6810 for UXG distribution. Conclusively, Table 10 and Figure 4 show that the UXG model has a very adequate fitting to the empirical data of the water capacity.

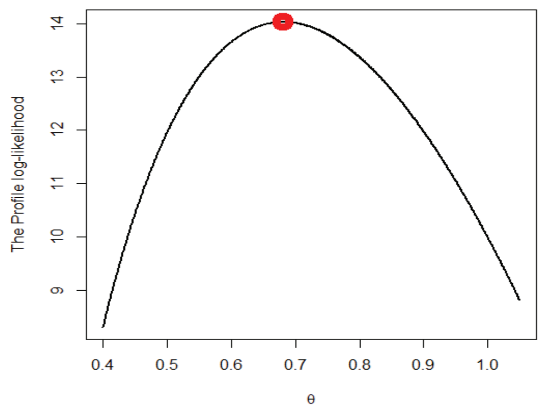

Fig. 5: Profile Log-Likelihood plot

7. CONCLUDING REMARKS

The UXG distribution is a new model introduced in this study. Some mathematical properties of the new distribution, such as its moments and associated measures, are derived. Two actuarial measures; VaR and TVaR are derived. For this model, we obtained a density of ordered statistics and two different characterizations. Comprehensive simulation studies on multiple sample sizes; small, moderate, and large sizes are used to compare the efficiency of the five methods of estimate stated above. The simulation results showed that the ML estimator is the best performing estimator for bias and MSE criteria for all sample sizes. UXG distribution presents a better fit to the water capacity data than other competent models. Thus we can say that the UXG distribution being a parsimonious model provides adequate and preferable modeling performance for water capacity data and is more flexible than some well-known probability distributions that are extensively famous for the application of lifetime data sets.

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