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Research Article

Mathematical Analysis on Conducting Sphere Embedded in Non Integer Dimensional Space

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Abstract: We have derived an analytical solution in low frequency using the idea of a fractional Laplacian equation. Fractional dimensional (FD) space has importance in describing the complex physics phenomena. Here, the Laplacian equation in spherical coordinated ($r,\theta,0$) is expressed in fractional dimensional space using Gegenbauer polynomials. The analytical solution is obtained by the separation variable method. The general solution is a product of angular and radial solutions and is independent of ϕ due to azimuthal symmetry. The classical solution is retained by setting fractional parameter α=3. Further, numerical results are discussed for different values of α and compared with available literature.

Keywords: Fractional dimensional space, Laplacian equation, Analytical solution, Separation variable method

1. INTRODUCTION

The non integer dimensional (NID)- space is applied to various areas of physics discussed by many researchers [1-20]. They have applied it accordingly, like Wilson [3] has already mentioned quantum field theory (QFT) in FD space. In addition, the FD space can be used as a parameter in the Ising limit of QFT [6]. [4] Stillinger has defined axiomatics basis for concept in modelling Schrodiner and Gibbsian's theory related to Wave and Statisitcs Mechanics in the non integer dimensional space. Svozil and Zellinger [10] have shown the basic concept of time dimension space, which provides a possible way to experimentally predict the time dimension space. It has already mentioned been that the fractional dimension space if the space time is less than 4 slightly. In the new era, Gauss Law {11] has introduced the fractional dimensional space. The solution to the electrostatic problems [13-18] have also been formulated in NID space.

In this article, we have solved a problem from [20]. The main focus is to apply the Laplacian equation to find electric potential and electric field induced due to a conducting sphere in FD Space. In this paper, we analyze the model of the problem and then solve the Laplacian equation for non integer dimensional space by the method of separation variable using low frequency. We find here two kinds of second order differential equations, one deals with radial part and other with angular part. Further, we investigate a general solution in FD space. Next, we construct a solution for the outside and inside the sphere and impose boundary conditions for the proposed regions. After solving the boundary value problem (BVP), we calculate the unknown coefficients, which leads us to find the electric potential and electric field by substituting known coefficients in FD Space. On the last we discuss numerical results and concluding remarks.

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2. MATHEMATICAL MODEL

We find here the Scalar Potential Ψ of a conducting sphere in a uniform external field shown in Fig1., of conductivity σ_2 and is buried in a host medium of conductivity σ_1. Since no time dependency is involved, therefore, the electric field is

$$E = -\nabla\Psi \tag{1}$$

we can determine the magnetic field from the Second Maxwell's equation

$$\nabla \times H = J \tag{2}$$

Let E_0 be the strength of the external field along x-axis.

$$E_0 = -\frac{\partial\Psi_0}{\partial x} \tag{3}$$

Negletting the constant of integration, we get

$$\Psi_0 = -E_0 x \tag{4}$$

To solve this problem for fractional dimensional space, we use the laplacian equation in non integer dimensional space [16] and [17]:

$$\nabla^2\Psi(r,\theta) = \left(\frac{\partial^2}{\partial r^2} + \frac{\alpha-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\sin^{\alpha-2}\theta}\frac{\partial}{\partial\theta}\sin^{\alpha-2}\theta\frac{\partial}{\partial\theta}\right)\Psi(r,\theta) = 0 \tag{5}$$

Eq(5) is separable and suppose

$$\Psi(r,\theta) = R(r)\Theta(\theta) \tag{6}$$

The obtained angular and radial differential equations [18] are

$$\left[\frac{d^2}{d\theta^2} + (\alpha-2)\cot\theta\frac{d}{d\theta} + l(l+\alpha-2)\right]\Theta(\theta) = 0 \tag{7}$$

$$\left[\frac{d^2}{dr^2} + \frac{\alpha-1}{r}\frac{d}{dr} + \frac{l(l+\alpha-2)}{r^2}\right]R(r) = 0 \tag{8}$$

The solution of the angular equation (7) is Gegenbauer polynomials in $cos\theta$ as explained in [16], namely

$$\Theta(\theta) = P_l^{\alpha/2-1}(\cos\theta),\ l = 0,1,2\ldots \tag{9}$$

which obeys the orthogonality relation:

$$\int_0^\pi P_l^{\alpha/2-1}(\cos\theta)P_{l'}^{\alpha/2-1}(\cos\theta)\sin^{\alpha-2}\theta d\theta = N(l)\delta_{l,l'} \tag{10}$$

$$N(l) = \frac{2^{(3-\alpha)}\pi\Gamma(l+\alpha-2)}{l!(l+\alpha/2-1)[\Gamma(l+\alpha/2-1)]^2} \tag{11}$$

The forms of first few Gegenbauer polynomials are

$$P_0^{\alpha/2-1}(z) = 1 \tag{12}$$

$$P_1^{\alpha/2-1}(z) = (\alpha-2)z \tag{13}$$

$$P_2^{\alpha/2-1}(z) = (\alpha/2-1)(\alpha z^2-1) \tag{14}$$

From Eq(6), the radial differential equation gives the first few solutions, such as

$$R_1(r) = r^l \tag{15}$$

$$R_2(r) = \frac{2}{r^{l+\alpha-2}} \tag{16}$$

The general solution of the Laplacian equation in Spherical Coordinates independent of ϕ can be written as

$$\Psi(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+\alpha-2}}\right)P_l^{\alpha/2-1}(\cos\theta),\ 2 < \alpha \leq 3) \tag{17}$$

$$\sum_{l=0}^{\infty} A_l r^l P_l^{\alpha/2-1}(\cos\theta) = \left[-E_0 r + \frac{B_1}{r^{\alpha-1}}\right]P_1^{\alpha/2-1}(\cos\theta),\ for\ r = R$$

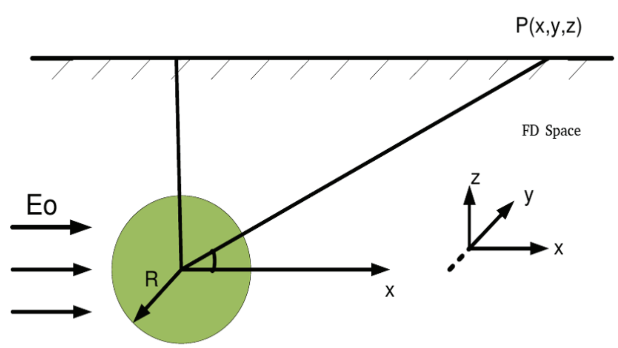

Fig. 1: Conducting Sphere Placed in Fraction Space

The external potential due to the sphere should be convergent at infinity. Inside the sphere, the solution must be bounded at origin. Hence the general solution reduces to following equations.

$$\Psi_e^{\alpha}(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+\alpha-2}} P_l^{\frac{\alpha}{2}-1}(\cos\theta),\ for\ r > 0 \quad (18)$$

$$\Psi_i^{\alpha}(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l^{\alpha/2-1}(\cos\theta),\ for\ r < 0 \quad (19)$$

These are known as the anomalous potentials. The physically the solution must be finite at the origin, it means $B_l = 0$ inside the sphere and out side the sphere at infinity it means $a_l = 0$.

At infinity, the normal potential can be expressed as

$$-E_0 x = -E_0 r(\alpha-2)\cos\theta = -E_0 r P_1^{\alpha/2-1}(\cos\theta) \quad (20)$$

Now, that external potential can be expressed as

$$\Psi_e(r,\theta) = \left[-E_0 r + \frac{B_1}{r^{\alpha-1}}\right] P_1^{\alpha/2-1}(\cos\theta),$$
$$for\ r > 0 \quad (21)$$

where $P_1^{\alpha/2-1}(\cos\theta) = (\alpha-2)\cos\theta$, while $-E_0 r P_1^{\alpha/2-1}(\cos\theta)$ and $B_1/r^{\alpha-1}$ shows the potential of the induced field respectively. At the surface of the sphere, both the potential and the normal component of the current density $n.J = \sigma E$ should be continous. So the potentials must be continuous at $r = R$.

$$\Psi_i^{\alpha} = \Psi_e^{\alpha} \quad (22)$$

$$\sum_{l=0}^{\infty} A_l r^l P_l^{\alpha/2-1}(\cos\theta) = \left[-E_0 r + \frac{B_1}{r^{\alpha-1}}\right] P_1^{\alpha/2-1}(\cos\theta),\ for\ r = R \quad (23)$$

Therefore, $A_l = 0$, all l except $l = 1$, we find

$$A_1 r P_1^{\frac{\alpha}{2}-1}(\cos\theta) = \left[-E_0 r + \frac{B_1}{r^{\alpha-1}}\right] C_1^{\frac{\alpha}{2}-1}$$
$$(\cos\theta),\ for\ r = R \quad (24)$$

$$A_1 R(\alpha-2)\cos\theta = \left[-E_0 R + \frac{B_1}{R^{\alpha-1}}\right](\alpha-2)\cos\theta \quad (25)$$

By comparing the above equation

$$A_1 = -E_0 + \frac{B_1}{R^{\alpha}} \quad (26)$$

Now the internal field becomes as

$$\Psi_i^{\alpha} = \left(-E_0 + \frac{B_1}{R^{\alpha}}\right) r(\alpha-2)\cos\theta,\ for\ r < 0 \quad (27)$$

Simply the obtained expression is

$$\Psi_i^{\alpha} = \left(-E_0 + \frac{B_1}{R^{\alpha}}\right) x,\ r < 0 \quad (28)$$

The strength of the electric field within the sphere is

$$E_i^{\alpha} = \left(E_0 - \frac{B_1}{R^{\alpha}}\right),\ r < 0 \quad (29)$$

Next, we introduce the conductivities of the medium by using boundary conditions for the continuity of the normal component of the current density J on the surface of the sphere, as

$$n.J_1 = n.J_2 \quad (30)$$

$$-\sigma_2 \frac{\partial \Psi_i}{\partial r} = -\sigma_1 \frac{\partial \Psi_e}{\partial r},\ r = R \quad (31)$$

$$\sigma_2\left(E_0 - \frac{B_1}{R^{\alpha}}\right)(\alpha-2)\cos\theta = \sigma_1\left(E_0 + \frac{(\alpha-1)B_1}{R^{\alpha}}\right)$$
$$(\alpha-2)\cos\theta \quad (32)$$

$$B_1 = \frac{\sigma_2-\sigma_1}{\sigma_2+(\alpha-1)\sigma_1} E_0 R^{\alpha} \quad (33)$$

For the integer order case $\alpha = 3$, we find

$$B_1 = \frac{\sigma_2-\sigma_1}{\sigma_2+2\sigma_1} E_0 R^3 \quad (34)$$

Hence, the external postential is

$$\Psi_e(r,\theta) = \left[-E_0 r + \frac{\sigma_2-\sigma_1}{\sigma_2+(\alpha-1)\sigma_1} E_0 R^{\alpha} \frac{1}{r^{\alpha-1}}\right](\alpha-2)\cos\theta \quad (35)$$

For the integer order case $\alpha = 3$, we see

$$\Psi_e(r,\theta) = \left[-E_0 r + \frac{\sigma_2-\sigma_1}{\sigma_2+2\sigma_1} E_0 R^3 \frac{1}{r^2}\right]\cos\theta \quad (36)$$

Where $E_0\ is\ the$ applied field that leads to the normal potential $\Psi_e^n = -r(\alpha-2)\cos\theta$ $and\ the$ anomalous potential is

$$\phi_e(r,\theta) = \frac{\sigma_2-\sigma_1}{\sigma_2+2\sigma_1} E_0 R^3 \frac{1}{r^{\alpha-1}}(\alpha-2)\cos\theta \quad (37)$$

In the geophysical perspective, it is important to consider the anomalous field only and to find the gradient of the potential such that

$$E = -\nabla\phi_e(r,\theta) = \frac{\sigma_2-\sigma_1}{\sigma_2+2\sigma_1} E_0 R^3 \frac{1}{r^{\alpha-1}}(\alpha-2)\cos\theta \quad (38)$$

$$E = -\nabla\phi_e(r,\theta) = \frac{\sigma_2-\sigma_1}{\sigma_2+2\sigma_1}E_0R^3(\alpha-2)$$

$$\left[\frac{(\alpha-1)x^2-y^2-z^2}{r^{\alpha+2}}i + \frac{\alpha xy}{r^{\alpha+2}}j + \frac{\alpha xz}{r^{\alpha+2}}k\right] \quad (39)$$

Normally we are bound to find the applied field, i.e. along the x- direction, we require only the quantity

$$P = \frac{\sigma_2-\sigma_1}{\sigma_2+(\alpha-1)\sigma_1}E_0R^\alpha(\alpha-2) \quad (40)$$

This is the electric dipole moment induced due to the sphere.

and

$$E = \frac{\sigma_2-\sigma_1}{\sigma_2+(\alpha-1)\sigma_1}E_0R^\alpha(\alpha-2)\left[\frac{(\alpha-1)x^2-y^2-z^2}{r^{\alpha+2}}u_x + \frac{\alpha xy}{r^{\alpha+2}}u_y + \frac{\alpha xz}{r^{\alpha+2}}u_z\right] \quad (41)$$

This is the secondary electric field intensity of the conducting sphere.

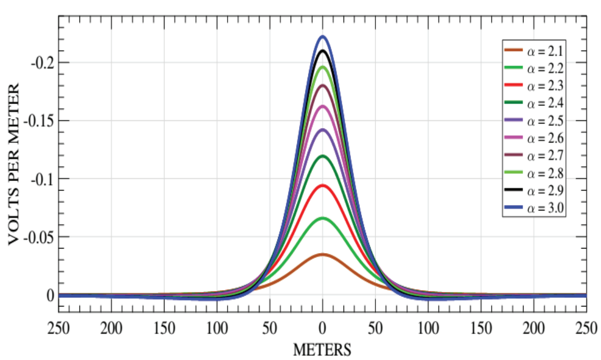

Fig.2: Graphical Representation of the Magnitude of the electric Field versus Distance x in Fractional Space electric field [20].

3 RESULTS AND DISCUSSION

The behaviour of the electric field E can be visualized by the variation in distance x (-250-250), from the figure 2. Through the graphical illustration, we observe that the field behaviour changes accordingly for the different values of FD space parameter ($\alpha = 2.1,2.2,2.3,2.4,2.5,2.6,2.7,2.8,2.9,3.0$). However, the Field magnitude behaves like ordinary amplitude [20] at ($\alpha = 3$). It is clear that the electric field depends on the FD space parameter. This behaviour predicts that the essentially the parameter α alters the material of the sphere due to which the different values of α cause the field amplitude variation, therefore, it affects the resonance conditions.

4. CONCLUSION

In the article, we have solved the Laplace equation for NID- space. The electric potential and electric field produced due to the conducting sphere are obtained in α -dimensional fractional space. We s

ee that the graphical behaviour of the curve shows that till $\alpha = 3$, the magnitude of the field increases by increasing the fractional parameter α between 2 and 3. From the results and discussion, we can say the field magnitude effects due to different FD space parameter values. Finally we can conclude that for all calculated results $\alpha = 3$, the classical results are retrieved.

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