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Research Article

Minimal Circular Nearly Strongly Balanced Repeated Measurements Designs in Unequal Period Sizes

H. M. Kashif Rasheed[1], Hafsa Khan[1], Rashid Ahmed[1], and Farrukh Jamal[1*]

[1]Department of Statistics, The Islamia University of Bahawalpur, Bahawalpur 61300, Pakistan

Abstract: In this article, some series are developed to generate circular nearly strongly balanced repeated measurements designs in periods of three different sizes when p_3 (smallest period size) = 2 through method of cyclic shifts (Rule II). These designs with varied period sizes have diversified applications, e.g. medicine, pharmacology, animal sciences and psychology. These designs are mainly used to balance the first order residual effect. Our proposed designs possess at least 98% efficiency of separability, therefore these designs are highly efficient to estimate the residual effects and direct effects independently.

Keywords: RMDs, Balanced RMDs, Residual effects, Minimal designs.

1. INTRODUCTION

A repeated measurements design (RMD) is strongly balanced with respect to the first-order residual effects if each treatment is immediately preceded λ' times by each other treatment (including itself). For given v and p, a balanced RMD is minimal if $\lambda' = 1$. RMDs are very useful in the fields of medicine, pharmacology, animal sciences and psychology. However, residual effects may arise in RMDs which is the major source of bias in the estimation of direct treatment effects. Strongly balanced (SBRMDs) are useful to estimate the direct effects and residual effects independently, therefore, these designs possess 100% efficiency of separability. These designs can be constructed through the method of cyclic shifts (Rule I) for some combinations of v and ps. For the remaining cases, the method of cyclic shifts (Rule II) generates nearly strongly balanced RMDs. RMD is nearly strongly balanced if each treatment is immediately preceded λ' times by each other treatment (including itself) except v-1 which does not preceded by itself. The reference [1] introduced and constructed balanced minimal RMDs for v even with $p = v$, where p is period size and v is the number of treatments. The reference [2] described the importance of RMDs in experiments of biology. The reference [3] showed that it is impossible to construct a balanced minimal RMD based on a cyclic group when v is odd. The reference [4] gave an easy method of constructing Latin square designs balanced for the immediate residual and other order effects. The reference [5] mentioned the importance of RMDs with unequal sizes of periods in industrial and agricultural experiments. The reference [6] introduced a model for RMDs with a circular structure of the residual effects. He proved the universal optimality of circular balanced uniform designs over a subclass of the possible designs. The reference [7] gave the complete solution of cyclic minimal balanced in linear periods for every possible $p < v$, under the two divisibility conditions (i) v/n and $(p\text{-}1)/(v\text{-}1)$. The reference [8] constructed circular balanced uniform RMD whenever the v is a prime or an even number. The reference [9] presented a simple method for the construction of circular balanced uniform RMD whenever the v is an odd number. The reference [10] constructed minimal CBRMDs with unequal period sizes. The reference [11] constructed BRMDs and SBMDs with equal and unequal period sizes through the method of cyclic shifts. The reference [12] introduced a general strategy to construct balanced RMDs for odd v along with

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*Corresponding Author: Farrukh Jamal <drfarrukh1982@gmail.com>

their analysis. The reference [13] constructed some CSBRMDs in periods of equal sizes. The reference [14] constructed some minimal strongly balanced changeover designs with first residuals. The reference [15] constructed some circular first- and second-order balanced RMDs through the method of cyclic shifts. The reference [16] presented CBRMDs for $p = 3$. The reference [17] constructed CBRMDs in periods of equal sizes for $p \leq v$. Some of their designs are minimal. The reference [18] developed some generators to generate MCBRMDs in periods of two different sizes. The reference [19] developed some generators to obtain MCSBRMDs in periods of three different sizes. The reference [20] constructed MCBRMDs in periods of three different sizes. The reference [21] developed some generators to obtain non-circular balanced and strongly balanced RMDs for periods of (i) equal, (ii) two, and (iii) three different sizes, through method of cyclic shifts. The reference [22] constructed MCSBRMDs in periods of unequal sizes. Even then MCSBRMDs could not be constructed for most of the combinations for v and p. This problem is resolved by constructing minimal circular nearly SBRMDs (MCNSBRMDs). In this article, some infinite series are developed to generate the MCNSBRMDs in periods of three different sizes when $p_3 = 2$, where $2 < p_2 \leq 11$ and $p_2 < p_1$.

The rest of the paper is organized as follows: In Section 2, Rule II is explained briefly to generate the CNSBRMDs in periods of three different sizes. In Section 3, the efficiency of separability is described. In section 4, some infinite series are developed through Rule II to obtain MCNSBRMDs for $p_3 = 2$. These series are very useful for researchers and experimenters. They can get the required designs just by putting the values needed for the series.

2. METHOD OF CYCLIC SHIFTS

Rule II is explained here briefly only for the construction of CNSBRMDs. For detail, see [13] and [15].

Rule II: Let $S_1 = [q_{11}, q_{12}, \ldots, q_{1(p_1-1)}]$, $S_2 = [q_{21}, q_{22}, \ldots, q_{2(p_2-1)}]$ and $S_3 = [q_{31}, q_{32}, \ldots, q_{3(p_3-2)}]$t be sets of shifts, where $0 \leq q_{ij} \leq v$-2. If each of 0, 1, …, v-2 appears λ' times S*, where S* = $[q_{11}, q_{12}, \ldots, q_{1(p_1-1)}, q_{21}, q_{22}, \ldots, q_{2(p_2-1)}, q_{31}, q_{32}, \ldots, q_{3(p_3-1)}, v\text{-}1\text{-}(q_{11}+ q_{12}+ \ldots+ q_{1(p_1-1)}) \bmod (v\text{-}1), v\text{-}1\text{-}(q_{21}+ q_{22}+ \ldots+ q_{1(p_2-1)}) \bmod (v\text{-}1)]$ then it will be CNSBRMD in periods of sizes p_1, p_2 & p_3. If $\lambda' = 1$ then it will be MCNSBRMD (MCNSBRMD).

Example 2.1: Set of shifts $S_1 = [1, 3, 2, 9, 8]$, $S_2 = [5, 0]$t, $S_3 = [6]$ provide following MCNSBRMD for $v = 11$ in $p_1 = 6$, $p_2 = 4$ & $p_3 = 2$.

3. EFFICIENCY

The reference [23] derived the following formula for Separability (Es) efficiency, which is also useful for our proposed MCNSBRMDs.

$$Es = \left[1 - \left\{ \frac{(l_1 + 4l_2)v - (l_1 + 2l_2)^2}{(v-1)(l_1 + 2l_2)^2} \right\}^{\frac{1}{2}} \right] \times 100\%$$

Where,

l_1: No. of a treatment immediately preceded by other treatment single time.

l_2: No. of a treatment immediately preceded by other treatment two times.

According to this criterion, our proposed designs possess at least 98% efficiency and are highly efficient.

4. SERIES TO GENERATE MCNSBRMDs

In this section, some series are developed by Rule II to generate MCNSBRMDs. In the following series, S* contains all values from 0, 1, …,v-2 exactly once, therefore, all series generate MCNSBRMD.

Series 4.1: If $v = 2mi+5$, $p_1 = 2m$, $m > 2$ integer, $p_2 = 4$ & $p_3 = 2$ then MCNSBRMD can be constructed through the following i+2 sets of shifts.

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}
0_3	1_4	2_5	3_6	4_7	5_8	6_9	7_0	8_1	9_2
1_0	2_1	3_2	4_3	5_4	6_5	7_6	8_7	9_8	0_9
4_1	5_2	6_3	7_4	8_5	9_6	0_7	1_8	2_9	3_0
6_4	7_5	8_6	9_7	0_8	1_9	2_0	3_1	4_2	5_3
5_6	6_7	7_8	8_9	9_0	0_1	1_2	2_3	3_4	4_5
3_5	4_6	5_7	6_8	7_9	8_0	9_1	0_2	1_3	2_4
B_{11}	**B_{12}**	**B_{13}**	**B_{14}**	**B_{15}**	**B_{16}**	**B_{17}**	**B_{18}**	**B_{19}**	**B_{20}**
0_{10}	1_{10}	2_{10}	3_{10}	4_{10}	5_{10}	6_{10}	7_{10}	8_{10}	9_{10}
5_0	6_1	7_2	8_3	9_4	0_5	1_6	2_7	3_8	4_9
5_5	6_6	7_7	8_8	9_9	0_0	1_1	2_2	3_3	4_4
10_5	10_6	10_7	10_8	10_9	10_0	10_1	10_2	10_3	10_4

B_{21}	B_{22}	B_{23}	B_{24}	B_{25}	B_{26}	B_{27}	B_{28}	B_{29}	B_{30}
0_6	1_7	2_8	3_9	4_0	5_1	6_2	7_3	8_4	9_5
6_0	7_1	8_2	9_3	0_4	1_5	2_6	3_7	4_8	5_9

$S_{j+1} = [mj+1, mj+2, \ldots, mj+m, v-mj-2, v-mj-3, \ldots, v-mj-m]$; $j = 0, 1, \ldots, i-1$.

$S_{i+1} = [0, (v-1)/2]t$, $S_{i+2} = [(v+1)/2]$

Example 4.1: MCNSBRMD is constructed through the following three sets of shifts for $v = 11$, $p_1 = 6$, $p_2 = 5$ & $p_3 = 2$.

$S_1 = [1, 2, 3, 9, 8]$ $S_2 = [0, 5]t$ $S_3 = [6]$

Series 4.2: If $v = 2mi+5$, $p_1 = 2m$, $m > 2$ integer, $p_2 = 5$ & $p_3 = 2$ then MCNSBRMD can be constructed through the following $i+2$ sets of shifts.

$S_{j+1} = [mj+1, mj+2, \ldots, mj+m, v-mj-2, v-mj-3, \ldots, v-mj-m]$; $j = 0, 1, \ldots, i-1$.

$S_{i+1} = [0, (v-3)/2, (v+1)/2]t$

$S_{i+2} = [(v-1)/2](1/2)$

Example 4.2: MCNSBRMD is constructed through the following three sets of shifts for $v = 11$, $p_1 = 6$, $p_2 = 5$ & $p_3 = 2$.

$S_1 = [1, 2, 3, 9, 8]$ $S_2 = [0, 4, 6]t$ $S_3 = [5](1/2)$

Series 4.3: If $v = 2mi+6$, $p_1 = 2m$, $m > 3$ integer, $p_2 = 5$ & $p_3 = 2$ then MCNSBRMD can be constructed through the following $i+2$ sets of shifts.

$S_{j+1} = [mj+1, mj+2, \ldots, mj+m, v-mj-2, v-mj-3, \ldots, v-mj-m]$; $j = 0, 1, \ldots, i-1$.

$S_{i+1} = [0, (v-4)/2, (v+2)/2]t$
$S_{i+2} = [v/2]$

Example 4.3: MCNSBRMD is constructed through the following three sets of shifts for $v = 14$, $p_1 = 8$, $p_2 = 6$ & $p_3 = 2$.

$S_1 = [1, 2, 3, 4, 12, 11, 10]$ $S_2 = [0, 5, 8]t$ $S_3 = [7]$

Series 4.4: If $v = 2mi+7$, $p_1 = 2m$, $m > 3$ integer, $p_2 = 6$ & $p_3 = 2$ then MCNSBRMD can be

constructed through the following i+2 sets of shifts.

S_{j+1} = [mj+1, mj+2,…,mj+m, v-mj-2, v-mj-3, …, v-mj-m]; j = 0, 1, …, i-1.

S_{i+1} = [0, (v-5)/2, (v-1)/2, (v+3)/2]t
S_{i+2} = [(v+1)/2]

Example 4.4: MCNSBRMD is constructed through the following three sets of shifts for v = 15, p_1 = 8, p_2 = 7 & p_3 = 2.

S_1 = [1, 2, 3, 4, 13, 12, 11] S_2 = [0, 5, 7, 9]t S_3 = [8]

Series 4.5: If v = 2mi+8, p_1 = 2m, m > 4 integer, p_2 = 7 & p_3 = 2 then MCNSBRMD can be constructed through the following i+2 sets of shifts.

S_{j+1} = [mj+1, mj+2,…,mj+m, v-mj-2, v-mj-3, …, v-mj-m]; j = 0, 1, …, i-1.

S_{i+1} = [0, (v-6)/2, (v-4)/2, (v+2)/2, (v+4)/2]t

S_{i+2} = [v/2]

Example 4.5: MCNSBRMD is constructed through the following three sets of shifts for v = 18, p_1 = 10, p_2 = 8 & p_3 = 2.

S_1 = [1, 2, 3, 4, 5, 16, 15, 14, 13]

S_2 = [0, 6, 7, 10, 11]t S_3 = [9]

Series 4.6: If v = 2mi+9, p_1 = 2m, m >4 integer, p_2 = 8 & p_3 = 2 then MCNSBRMD can be constructed through the following i+2 sets of shifts.

S_{j+1} = [mj+1, mj+2,…,mj+m, v-mj-2, v-mj-3, …, v-mj-m]; j = 0, 1, …, i-1.

S_{i+1} = [0, (v-7)/2, (v-5)/2, (v-1)/2, (v+3)/2, (v+5)/2]t

S_{i+2} = [(v+1)/2]t

Example 4.6: MCNSBRMD is constructed through the following three sets of shifts for v = 19, p_1 = 10, p_2 = 9 & p_3 = 2.

S_1 = [1, 2, 3, 4, 5, 17, 16, 15, 14]

S_2 = [0, 6, 7, 9, 11, 12]t S_3 = [10]

Series 4.7: If v = 2mi+10, p_1 = 2m, m > 5 integer, p_2 = 9 & p_3 = 2 then MNSBRMD can be constructed through the following i+2 sets of shifts.

S_{j+1} = [mj+1, mj+2,…,mj+m, v-mj-2, v-mj-3,

…, v-mj-m]; j = 0, 1, …, i-1.

S_{i+1} = [0, (v-8)/2, (v-6)/2, (v-4)/2, (v+2)/2, (v+4)/2, (v+6)/2]t

S_{i+2} = [v/2]

Example 4.7: MCNSBRMD is constructed through the following three sets of shifts for v = 22, p_1 = 12, p_2 = 10 & p_3 = 2.

S_1 = [1, 2, 3, 4, 5, 6, 20, 19, 18, 17, 16]
S_2 = [0, 7, 8, 9, 12, 13, 14]t
S_3 = [11]

Series 4.8: If v = 2mi+11, p_1 = 2m, m > 5 integer, p_2 = 10 & p_3 = 2 then MCNSBRMD can be constructed through the following i+2 sets of shifts.

S_{j+1} = [mj+1, mj+2,…,mj+m, v-mj-2, v-mj-3, …, v-mj-m]; j = 0, 1, …, i-1.

S_{i+1} = [0, (v-9)/2, (v-7)/2, (v-5)/2, (v-1)/2, (v+3)/2, (v+5)/2, (v+7)/2]t S_{i+2} = [(v+1)/2]t

Example 4.8: MCNSBRMD is constructed through the following three sets of shifts for v = 23, p_1 = 12, p_2 = 11 & p_3 = 2.

S_1 = [1, 2, 3, 4, 5, 6, 21, 20, 19, 18, 17]
S_2 = [0, 7, 8, 9, 11, 13, 14, 15]t
S_3 = [12]

Designs constructed through series 4.1 for $v \leq 100$ are presented as Appendix.

5. CONSTRUCTION OF MCNSBRMDs IN PERIOD OF SIZES P_1, P_2 WITH P_3 = 2

Series 5.1: MCNSBRMDs can be constructed with p_1 = 5, p_2 = 3 and p_3 = 2 for v = 5i+4; i odd through the following i sets of shifts for p_1 and one set each for p_2 and p_3.

S_j = [q_{j1}, q_{j2}, q_{j3}, q_{j4}]; j = 1, 2, …, i.

$S_{i+1} = [(v\text{-}3)/2]$; $S_{i+2} = [(v\text{-}1)/2]$t; Where

- $0 \leq q_{j1}, q_{j2}, q_{j3}, q_{j4} \leq v\text{-}2$ but $\neq (v\text{-}3)/2, (v\text{-}1)/2, (v+1)/2$.
- S* contains each of 0, 1, 2, …, v-2 exactly once.
- $S^* = [q_{j1}, q_{j2}, q_{j3}, q_{j4}, v\text{-}1\text{-}(q_{j1}+q_{j2}+q_{j3}+q_{j4}), (v\text{-}3)/2, (v\text{-}1)/2, (v+1)/2]$

6. RESULTS AND DISCUSSION

MCSBRMDs are 100% efficient to estimate the residual effects and direct effects independently. MCSBRMDs cannot be constructed for each and every combination of v and p. Our proposed MCNSBRMDs possess at least 98% efficiency of separability, therefore, these designs are best alternates to the MCSBRMDs for the situations where MCSBRMDs cannot be developed.

Future work will develop an algorithm coded with R-language to generate MCNSBRMDs in periods of equal and unequal sizes. Proposed designs will be applied in the field of medicine and animal sciences to get the numerical results.

7. CONFLICT OF INTEREST

There is no conflict of interest.

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Appendix

v	p_1	p_2	p_3	Sets of Shifts	ES
9	5	3	2	[1,2,6,7]+[3]+[4]t	0.96
19	5	3	2	[1,2,3,12]+[5,6,7,14]+[11,15,16,17]+[10]+[9]t	0.93
29	5	3	2	[19,2,3,4]+[6,17,8,27]+[10,11,12,16]+[1,20,23,22]+[24,25,5,9]+[15]+[14]t	0.99
39	5	3	2	[1,2,3,4]+[6,33,8,24]+[11,12,13,14]+[16,17,21,22]+[9,25,30,27]+ [29,15,31,32]+[34,35,36,37]+[20]+[19]t	0.99
49	5	3	2	[1,2,3,4]+[37,7,8,26]+[40,11,31,14]+[32,17,13,19]+[21,22,5,28]+ [42,27,30,29]+[33,34,35,36]+[39,9,41,43]+[10,45,46,47]+[25]+[24]t	0.99
59	5	3	2	[1,2,3,4]+[6,7,31,9]+[11,12,13,14]+[16,17,18,19]+[21,22,23,24]+ [41,27,53,33]+[34,35,36,37]+[39,45,10,42]+[43,44,40,47]+ [49,55,51,52]+[54,50,56,57]+[30]+[29]t	0.99
69	5	3	2	[1,2,3,4]+[6,5,10,9]+[11,12,13,14]+[16,24,8,21]+[19,22,25,17]+ [26,29,28,30]+[31,32,36,37]+[39,27,41,40]+[20,48,44,49]+[45,47,50,55]+ [54,52,56,59]+[46,60,61,63]+[62,64,65,66]+[35]+[34]t	0.99
79	5	3	2	[1,2,3,4]+[6,7,8,9]+[18,12,13,14]+[16,17,11,19]+[20,10,50,71]+ [25,27,28,29]+[31,26,35,34]+[36,33,67,42]+[44,77,24,46]+[54,22,51,45]+ [53,55,37,57]+[59,60,61,74]+[76,64,65,66]+[69,70,23,72]+[49,75,52,63]+ [40]+[39]t	0.99
89	5	3	2	[1,2,3,4]+[59,7,8,9]+[11,12,13,87]+[16,17,21,19]+[18,22,35,67]+ [46,27,28,51]+[31,32,33,20]+[36,37,38,39]+[62,49,54,47]+[41,58,40,77]+ [25,55,29,57]+[50,23,61,42]+[63,64,65,66]+[69,70,73,72]+[74,75,76,56]+ [79,80,81,82]+[83,84,85,86]+[45]+[44]t	0.99
99	5	3	2	[1,2,3,4]+[6,7,8,9]+[11,12,51,14]+[16,17,18,19]+[30,21,22,25]+ [26,27,91,29]+[31,32,33,34]+[36,37,38,39]+[55,41,42,43]+[35,47,13,56]+ [54,40,52,95]+[24,60,61,86]+[64,65,20,67]+[69,77,71,72]+[74,75,80,90]+ [83,76,81,82]+[84,85,87,57]+[89,94,58,92]+[62,44,96,97]+[50]+[49]t	0.99
11	7	3	2	[2,3,7,8,9]+[6]+[5]t	0.86

v	p_1	p_2	p_3	Sets of Shifts	ES
25	7	3	2	[1,2,3,4,5,6]+[8,9,10,14,15,16]+[18,19,20,1,22,23]+[13]+[12]t	0.92
39	7	3	2	[1,2,3,4,5,6]+[14,9,10,11,12,13]+[8,15,16,30,22,23]+[25,26,27,28,29,31]+ [32,33,34,35,36,37]+[20]+[19]t	0.99
53	7	3	2	[1,2,3,4,5,6]+[7,9,11,37,12,13]+[51,16,17,18,19,21]+[22,23,24,8,29,30]+ [32,33,34,35,36,38]+[39,40,41,42,43,45]+[44,46,47,48,49,50]+[27]+[26]t	0.99
67	7	3	2	[1,20,51,4,5,6]+[8,9,10,11,12,13]+[14,38,16,17,26,19]+ [22,23,24,25,18,65]+[29,30,31,7,36,37]+[39,40,41,42,43,44]+ [46,47,48,49,50,35]+[52,53,54,56,57,58]+[59,60,61,62,63,64]+[34]+[33]t	0.99
81	7	3	2	[1,2,3,4,5,6]+[8,9,10,11,12,13]+[14,15,16,7,18,70]+ [21,22,23,24,25,26]+[45,30,31,32,33,34]+[36,37,38,42,43,44]+ [46,47,48,49,50,51]+[53,60,55,56,66,58]+[54,61,62,63,64,68]+ [67,65,69,79,71,72]+[73,74,75,76,77,78]+[41]+[40]t	0.99
95	7	3	2	[72,2,25,4,5,6]+[8,9,10,11,12,37]+[15,85,17,18,19,20]+[22,23,44,3,26,27] + [29,30,31,32,33,34]+[36,53,38,39,40,41]+[21,24,45,49,50,51]+ [13,54,55,56,57,58]+[60,61,62,63,64,65]+[67,68,69,70,71,59]+ [73,87,76,77,78,79]+[81,82,52,84,16,75]+[86,88,89,90,91,92]+[48]+[47]t	0.99
13	9	3	2	[1,2,3,4,8,9,10,11]+[5]+[6]t	0.98
31	9	3	2	[24,1,2,3,4,5,6,7,8]+[9,10,26,12,13,23,18,19,20]+ [21,22,17,0,25,11,27,28,29]+[14]+[15]t	0.99
49	9	3	2	[1,2,3,4,5,6,7,8]+[10,11,18,13,14,15,16,17]+ [19,20,21,22,26,27,28,29]+[31,32,33,34,35,39,37,38]+ [40,41,42,43,44,45,46,47]+[25]+[24]t	0.99
67	9	3	2	[1,2,3,4,5,6,7,8]+[10,11,12,13,14,15,16,17]+[19,20,21,22,38,9,25,26]+ [27,28,29,31,35,36,37,41]+[40,23,48,43,65,45,46,47]+ [49,50,51,52,53,54,55,56]+[57,58,59,60,61,62,63,64]+[34]+[33]t	0.99
85	9	3	2	[1,2,3,4,5,6,7,8]+[10,11,12,13,14,15,16,17]+ [19,20,21,22,23,24,25,26]+[38,29,30,31,32,33,34,35]+ [56,28,39,40,83,45,46,47]+[49,50,70,52,53,54,55,37]+ [58,59,9,61,62,63,64,71]+[67,68,69,51,65,18,73,27]+ [75,76,77,78,79,80,81,82]+[43]+[42]t	0.98
15	5	4	2	[3,4,6,7]+[9,10,11,12]+[13]+[2,5]t	0.98
25	5	4	2	[9,4,5,6]+[8,3,11,12]+[7,15,16,21]+[19,20,17,22]+[23]+[2,10]t	0.99
35	5	4	2	[3,4,5,6]+[8,32,10,11]+[12,14,28,17]+[30,20,21,22]+[24,25,26,27]+ [29,19,13,18]+[33]+[2,15]t	0.99
45	5	4	2	[3,4,5,7]+[8,9,10,11]+[13,14,15,16]+[42,19,21,23]+[24,33,36,17]+ [29,12,31,32]+[37,34,35,26]+[38,39,40,41]+[43]+[2,20]t	0.99
55	5	4	2	[3,4,5,6]+[8,9,19,11]+[51,14,15,16]+[18,10,20,22]+[23,24,13,27]+ [29,30,31,32]+[33,52,35,42]+[39,28,41,37]+[44,45,46,47]+ [49,50,26,43]+[53]+[2,25]t	0.99
65	5	4	2	[3,4,5,6]+[8,9,10,11]+[32,14,50,20]+[62,19,16,24]+[23,21,25,37]+ [27,28,29,31]+[42,35,36,15]+[39,40,41,34]+[43,54,45,33]+ [49,47,51,52]+[44,55,56,48]+[58,59,60,61]+[63]+[2,30]t	0.97
75	5	4	2	[3,4,5,6]+[37,9,10,11]+[13,14,15,20]+[18,19,16,23]+[21,24,27,26]+ [28,29,30,61]+[33,39,36,8]+[43,58,41,42]+[44,45,46,47]+	0.99

v	p_1	p_2	p_3	Sets of Shifts	ES
				[49,22,51,52]+[62,55,17,57]+[59,60,53,54]+[64,65,66,67]+ [69,25,71,63]+[73]+[2,35]t	
85	5	4	2	[3,4,5,6]+[8,9,10,50]+[13,14,15,16]+[67,19,20,45]+[23,24,25,57]+ [28,29,32,31]+[33,34,35,36]+[12,22,41,49]+[46,52,21,77]+ [42,11,51,37]+[54,55,43,47]+[59,82,61,62]+[63,64,65,60]+ [69,70,71,58]+[74,75,76,38]+[79,80,81,18]+[83]+[2,40]t	0.99
95	5	4	2	[3,4,5,6]+[8,9,10,11]+[13,14,15,16]+[18,19,20,30]+[23,47,25,26]+ [28,24,29,46]+[33,34,35,54]+[31,39,40,41]+[42,86,44,89]+ [49,50,83,52]+[12,55,17,51]+[59,77,27,62]+[63,64,65,68]+ [80,87,71,72]+[74,75,0,60]+[43,92,81,82]+[84,85,79,70]+ [38,90,91,69]+[93]+[2,45]t	0.99
19	7	4	2	[3,4,5,6,8,10]+[11,12,13,14,15,16]+[17]+[2,7]t	0.99
33	7	4	2	[3,4,5,6,7,10]+[9,23,12,13,15,16]+[18,19,20,21,22,11]+[24,25,26,27,28,30]+ [31]+[2,14]t	0.99
47	7	4	2	[3,4,5,6,7,8]+[30,10,11,9,17,15]+[14,18,22,20,25,23]+ [19,26,27,28,29,31]+[32,44,34,35,36,37]+[39,40,41,42,43,33]+[45]+[2,21] t	0.99
61	7	4	2	[3,4,5,6,7,8]+[33,11,12,13,14,15]+[17,18,19,20,21,9]+[23,24,25,49,29,30] + [57,10,34,35,36,37]+[39,40,41,42,43,44]+[46,47,48,26,50,38]+ [53,54,55,56,58,32]+[59]+[2,28]t	0.99
75	7	4	2	[3,4,5,6,7,8]+[10,11,12,13,14,16]+[17,18,19,20,21,22]+[24,25,26,27,28,29]+ [15,32,38,34,36,37]+[39,40,55,42,43,44]+[46,47,48,9,50,51]+ [52,53,54,56,23,58]+[60,61,62,71,64,67]+[65,68,69,70,57,49]+[73]+[2,35] t	0.99
89	7	4	2	[3,4,5,6,7,8]+[10,11,12,13,18,15]+[17,14,19,20,21,23]+ [43,25,26,27,28,29]+[31,32,33,34,35,51]+[38,39,40,41,24,45]+ [46,47,80,49,50,36]+[53,54,22,56,57,58]+[60,61,63,68,64,65]+ [67,16,69,70,71,81]+[74,75,72,77,78,79]+[30,76,82,83,84,85]+[87]+[2,42] t	0.99
23	9	4	2	[3,16,5,6,7,8,10,11]+[13,14,15,4,17,18,19,20]+[21]+[2,9]t	0.99
41	9	4	2	[3,4,5,6,7,8,9,10]+[38,11,14,15,37,17,19,20]+ [21,22,23,24,25,26,27,32]+[30,31,13,33,34,35,36,16]+[39]+[2,18]t	0.99
59	9	4	2	[3,4,5,12,7,8,11,10]+[51,13,14,15,16,17,20,19]+ [21,38,23,24,25,26,28,29]+[30,31,32,33,34,35,36,37]+ [40,41,42,43,44,45,46,47]+[49,50,6,52,53,54,55,39]+[57]+[2,27]t	0.99
77	9	4	2	[3,4,5,6,7,8,9,10]+[12,13,14,15,16,17,18,19]+ [21,22,23,35,25,26,27,11]+[60,30,32,33,34,58,37,20]+ [29,71,42,43,74,45,46,47]+[49,50,51,52,53,54,55,44]+ [31,73,72,61,62,63,64,65]+[67,68,69,70,56,40,39,57]+[75]+[2,36]t	0.99
95	9	4	2	[3,4,5,6,7,8,9,10]+[12,13,14,15,16,17,37,28]+ [20,21,22,23,24,25,26,27]+[30,31,32,33,34,64,11,18]+ [75,40,41,35,43,44,54,47]+[49,50,51,52,53,46,55,66]+	0.99

v	p_1	p_2	p_3	Sets of Shifts	ES
				[58,59,60,61,62,63,79,65]+[67,68,69,70,71,82,73,74]+ [76,77,78,19,80,81,72,83]+[85,86,87,88,89,90,38,39]+[93]+[2,45]t	
13	7	5	2	[11,5,6,7,10,9]+[8]+[1,2,3]t	0.98
27	7	5	2	[15,4,5,6,7,24]+[9,11,12,13,14,3,16]+[25,19,20,21,22,23]+[18]+[1,2,10]t	0.99
41	7	5	2	[3,4,5,6,7,36]+[10,11,12,13,14,25]+[16,18,20,21,22,23]+ [24,15,26,27,28,31]+[39,33,34,30,38,37]+[32]+[1,2,17]t	0.99
55	7	5	2	[3,4,5,6,7,52]+[10,11,12,13,38,43]+[17,18,19,40,16,22]+ [20,26,15,28,29,21]+[51,33,34,25,36,37]+[39,9,41,42,44,27]+ [53,47,48,49,50,32]+[46]+[1,2,24]t	0.99
69	7	5	2	[19,4,5,6,7,66]+[34,11,12,13,14,15]+[16,58,18,3,20,21]+ [25,10,26,27,28,65]+[39,9,33,35,36,22]+[38,40,41,42,43,44]+ [46,47,48,49,54,51]+[52,53,50,55,56,57]+[67,61,62,63,64,32]+ [60]+[1,2,31]t	0.99
83	7	5	2	[3,4,5,6,7,80]+[20,11,12,13,14,15]+[17,18,60,10,21,22]+ [24,34,26,27,28,29]+[31,32,33,25,49,23]+[37,39,41,42,43,44]+ [46,47,54,35,50,51]+[71,48,55,56,57,58]+[36,61,62,63,64,52]+ [66,67,68,69,70,40]+[73,81,75,76,77,19]+[74]+[1,2,38]t	0.99
97	7	5	2	[3,4,5,6,7,94]+[10,11,12,13,14,15]+[17,18,19,20,80,22]+ [24,25,26,27,28,29]+[31,32,90,34,35,36]+[83,39,40,41,42,43]+ [46,47,48,81,50,52]+[53,54,55,56,57,58]+[70,61,62,63,64,65]+ [67,68,69,84,71,77]+[74,75,76,72,38,66]+[49,82,78,89,87,86]+ [59,37,23,91,92,93]+[88]+[1,2,45]t	0.99
15	9	5	2	[13,5,6,7,8,9,10,12]+[11]+[1,2,4]t	0.99
33	9	5	2	[3,4,5,6,7,8,9,30]+[12,14,15,16,17,18,19,20]+[21,31,23,25,26,27,28,11]+ [22]+[1,2,13]t	0.99
51	9	5	2	[13,4,5,6,7,8,9,48]+[38,3,14,15,16,46,18,19]+[21,23,24,25,11,27,28,29]+ [26,32,33,34,35,36,37,20]+[49,41,42,43,44,45,17,30]+[40]+[1,2,22]t	0.99
69	9	5	2	[3,4,5,6,7,8,9,66]+[12,13,14,15,16,17,18,20]+ [19,21,39,23,24,25,26,27]+[48,32,33,34,35,36,37,38]+ [40,41,30,43,44,45,46,29]+[49,50,65,52,53,54,55,56]+ [67,59,60,61,62,63,64,51]+[58]+[1,2,31]t	0.99
87	9	5	2	[65,5,74,6,7,8,9,84]+[12,83,14,15,16,17,18,19]+ [45,22,23,24,25,26,27,29]+[28,30,31,11,33,34,35,36]+ [39,41,42,43,44,21,4,47]+[49,50,13,52,53,54,55,56]+ [58,59,60,61,62,38,3,32]+[67,68,69,70,71,72,73,46]+ [85,77,78,79,80,81,82,51]+[76]+[1,2,40]t	0.99
21	7	6	2	[5,11,7,18,9,10]+[19,13,14,15,16,17]+[12]+[1,2,3,4]t	0.99
35	7	6	2	[10,5,6,7,32,9]+[12,24,14,15,16,17]+[18,19,20,21,22,23]+ [25,27,28,29,30,31]+[26]+[1,2,3,11]t	0.99
49	7	6	2	[4,5,6,7,46,9]+[11,12,13,14,20,16]+[17,37,21,22,23,24]+ [25,27,28,29,30,15]+[33,34,35,36,45,26]+[47,41,42,43,44,32]+ [40]+[1,2,3,18]t	0.99
63	7	6	2	[33,4,5,6,7,60]+[11,12,46,14,15,16]+[18,19,20,30,40,23]+ [26,27,28,29,21,31]+[59,34,35,45,37,38]+[22,41,42,43,44,17]+ [47,48,49,50,51,52]+[61,55,56,57,58,32]+[54]+[1,2,3,25]t	0.99

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v	p_1	p_2	p_3	Sets of Shifts	ES
77	7	6	2	[4,5,6,7,74,9]+[11,12,13,14,15,16]+[18,19,20,21,22,23]+ [24,26,27,28,17,30]+[51,34,35,36,37,38]+[40,65,42,43,44,31]+ [25,48,49,50,33,53]+[54,55,56,57,58,59]+[61,62,63,69,75,66]+ [67,52,64,70,10,72]+[68]+[1,2,3,32]t	0.99
91	7	6	2	[4,5,52,7,88,53]+[11,12,13,14,15,16]+[18,19,29,65,68,23]+ [25,26,27,28,20,30]+[73,33,34,35,36,37]+[58,41,42,43,44,45]+ [46,47,21,49,50,51]+[76,55,56,57,40,59]+[60,62,31,64,77,66]+ [63,69,70,71,72,38]+[75,54,80,78,79,10]+[89,83,84,85,86,32]+ [82]+[1,2,3,39]t	0.99
25	9	6	2	[4,5,7,8,9,16,11,12]+[23,15,22,17,18,19,20,21]+[14]+[1,2,3,6]t	0.99
43	9	6	2	[4,5,6,7,8,9,40,11]+[13,14,16,17,18,19,20,21]+ [23,24,25,26,27,28,29,39]+[41,33,34,35,0,37,38,12]+[32]+[1,2,3,15]t	0.99
61	9	6	2	[4,5,6,7,8,9,58,11]+[13,14,15,16,17,18,19,20]+ [22,38,25,26,27,28,29,45]+[32,33,34,35,36,37,23,39]+ [41,42,49,44,40,46,47,21]+[59,51,52,53,54,55,56,57]+[50]+[1,2,3,24]t	0.99
79	9	6	2	[4,5,6,7,8,9,76,41]+[13,14,15,28,17,21,16,20]+ [22,23,24,25,26,27,19,29]+[31,32,34,35,18,37,38,57]+ [11,42,67,44,45,46,47,48]+[50,51,52,53,66,55,56,36]+ [59,60,61,62,63,64,65,54]+[77,69,70,71,72,73,74,75]+[68]+[1,2,3,33]t	0.99
97	9	6	2	[4,5,6,7,8,9,94,11]+[13,74,15,16,17,28,19,22]+ [20,23,24,25,26,27,12,56]+[31,32,33,34,35,18,37,38]+ [40,41,43,44,0,46,47,57]+[50,93,52,53,54,55,29,45]+ [59,60,61,62,63,64,14,39]+[68,69,70,71,72,73,65,21]+ [77,78,79,80,81,82,83,36]+[85,95,87,88,89,90,91,92]+[86]+[1,2,3,42]t	0.99
17	9	7	2	[2,3,4,5,10,7,8,9]+[15]+[6,11,12,13,14]t	0.99
35	9	7	2	[2,3,4,5,6,7,8,9]+[11,12,13,14,15,16,17,18]+ [10,21,22,23,31,25,26,27]+[33]+[28,29,30,0,32]t	0.99
53	9	7	2	[2,3,4,5,6,40,8,9]+[11,12,13,14,15,16,7,18]+ [20,21,22,23,24,25,26,28]+[29,30,31,32,33,34,35,36]+ [38,39,17,41,42,43,10,45]+[51]+[46,47,48,49,44]t	0.99
71	9	7	2	[2,3,4,5,6,7,8,9]+[11,12,27,14,15,16,17,18]+ [19,21,63,68,24,44,13,28]+[29,30,31,32,33,34,35,36]+ [38,39,40,41,42,43,25,45]+[47,48,49,50,51,52,23,54]+ [56,57,58,59,60,61,62,22]+[69]+[64,65,66,67,53]t	0.99
89	9	7	2	[2,3,4,5,6,7,8,9]+[11,12,13,14,65,16,17,18]+ [15,20,21,22,23,24,25,26]+[29,30,73,54,33,34,35,36]+ [38,39,40,41,42,43,27,45]+[47,48,49,50,51,64,53,32]+ [56,57,58,59,60,61,86,72]+[81,66,67,68,78,70,71,63]+ [74,75,76,77,69,79,80,55]+[87]+[82,83,84,85,62]t	0.99
27	9	8	2	[2,3,4,5,6,7,16,9]+[11,24,13,14,15,8,17,18]+[25]+[19,20,21,22,23,12]t	0.99
45	9	8	2	[2,3,4,5,6,7,8,9]+[11,12,13,20,15,16,17,18]+[33,21,22,23,24,25,26,27]+ [29,41,31,32,42,34,35,36]+[43]+[37,38,39,40,30,14]t	0.99
63	9	8	2	[2,3,4,5,6,7,8,9]+[11,12,13,14,15,16,17,36]+[19,20,22,23,24,25,26,27]+ [29,30,31,32,33,34,35,58]+[38,39,40,41,42,43,44,48]+ [47,60,49,50,51,21,56,54]+[61]+[55,53,57,10,59,45]t	0.99

v	p_1	p_2	p_3	Sets of Shifts	ES
81	9	8	2	[2,3,4,5,6,7,8,9]+[11,12,13,14,15,16,17,18]+[20,21,22,23,24,58,26,27]+ [29,30,31,32,33,34,35,68]+[38,39,40,41,42,43,63,57]+ [46,47,48,49,51,52,53,54]+[56,45,10,59,60,61,62,72]+ [64,78,66,67,50,69,70,71]+[79]+[73,74,75,76,77,65]t	0.99
99	9	8	2	[2,3,4,5,6,7,8,9]+[11,12,13,14,15,16,17,88]+[20,21,37,19,24,71,26,27]+ [53,64,31,32,63,34,35,36]+[38,39,40,41,42,43,23,96]+ [47,48,45,50,51,52,28,25]+[56,57,58,59,60,61,62,22]+ [33,65,66,29,68,69,70,90]+[67,75,76,77,78,79,80,81]+ [83,84,85,86,87,18,89,72]+[97]+[91,92,93,94,95,74]t	0.99