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Research Article

Response of Homogeneous Conducting Sphere in Non-Integer Dimensional Space

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Abstract: In this paper, we have investigated electric potential and field analytically for homogeneous conducting sphere by solving the Laplacian equation in fractional dimensional space. The laplacian equation in fractional space describes complex phenomena of physics. The separation variable method is used to solve the Laplace differential equation. The mathematical formulae governing the interaction of a low-frequency source of electric current with a spherical anomaly are derived in fractional dimensional space. These formulae are used to determine the apparent resistivity and induced-polarization response. The potential due to the current point source in fractional space is derived using Gegenbauer polynomials. The electric field inrensity of the homogeneous conducting sphere is calculated using the electric potential due to a current point source outside the sphere. The results are compared analytically with classical results by setting the fractional parameter α=3.

Keywords: Laplacian Equation, Electric Potential, Fractional Dimensional Space, Separation Variable Method, Resistivity, Induced

1. INTRODUCTION

The idea of fractional-dimensional space (FD space) is novel and very useful in various disciplines of physics, and it has been discussed by many people [1-16]. As Wilson [3] discussed in FD space quantum field theory, they have applied it accordingly. Further, the FD space can be used as a parameter in the Ising limit of quantum field theory [6]. Stillinger [4] has provided an axiomatic basis for this concept for the formulation of Schrodinger wave mechanics and Gibbsian statistical mechanics in the α -dimensional space. Svozil and Zeilinger [10] have presented an operationalistic definition of the space time dimension, which provides the possibility of experimental determination of space time dimension. It has also been stated that the fractional dimension of space time is slightly less than 4. In the new era, Gauss law [11] has been formulated in α -dimensional fractional space. The solution of electrostatic problems [13-18], have also been investigated in the FD space "($2< \alpha \leq 3$)". The plan of the paper is as follows: In the first section, we construct an analytical solution for a conducting sphere in the presence of a point current source. This is accomplished in two steps: Firstly, a solution for the electric scalar potential, due to a point current source within a host medium of resistivity ρ, is solved in fractional dimensional space. This solution is then re-expressed within a polar coordinate system and decomposed into a sum of spherical harmonic modes in fractional dimensional space. Secondly, the solution for a conducting sphere within a host medium is determined by solving a boundary value problem for each spherical harmonic fractional space. For low-frequency probing, the solution of this physics problem is governed by Laplace's equation in fractional dimensional space. We concern ourselves with the variation in apparent resistivity using single- and cross-borehole probing.

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2. MATHEMATICAL MODEL

The sphere of radius a and resistivity ρ_1 is embedded in an infinite homogeneous medium of resistivity ρ. We consider the case where a source of electrical current of magnitude I is being injected into a host medium with resistivity ρ, at location $(x, y, z) = (x_0, 0,0)$. Assuming the medium is lossless, this results in a current density J which flows radially outwards from the source, with magnitude:

$$\chi_n = (\frac{b}{a})^{2n+1}\left[1 - \frac{(2n+(\alpha-2)(b^{\alpha-3}+\beta_n)}{(2n+(\alpha-2))+i\alpha_1\omega b^{\alpha-3}(1+\beta_n)}\right] \quad (1)$$

where

$$\beta_n = a^{\alpha-3}(\frac{a}{b})^{2n+1}\left[\frac{(n+\alpha-2)-\delta_n}{n+\delta_n}\right]$$

and

$$\delta_n = \left(\frac{\mu_0}{\mu_1}\right)\frac{I_n(\gamma a)}{\hat{I}_n(\gamma a)}$$

Further,

$$\hat{I}_n(\gamma a) = \sqrt{\frac{\pi\gamma a}{2}}\, I_n(\gamma a)$$

where R is the distance from the source to the point of measure P, and $4\pi R^2$ is the area of a ball centred at the source. Because our problem is electrostatic $E = -\nabla\phi$ according to Faraday's law. The scalar electric potential Ψ can be obtained by integrating the electric field from R to ∞. By substituting Ohm's law $E = \rho J$ into the path integral:

$$\int_R^\infty E.dl = \int_R^\infty \frac{\rho I}{4\pi r^2}dr = \frac{\rho I}{4\pi R} \quad (3)$$

For reasons which will become apparent in the next section, we would like to re-express Ψ in terms of a radial coordinate system (r, θ, π), centred at $(x, y, z) = (0,0,0)$. Because the points which represent the problem geometry do not necessarily form a right-triangle, R must be expressed using the cosine law: $R = (r^2 + x_0^2 - 2rx_0\cos\theta)^{1/2}$ For solutions where $r < x_0$, $1/R$ can be split into a sum of spherical harmonic modes using the binomial theorem:

$$\frac{1}{R} = \frac{1}{x_0}\left[1 + (\frac{r}{x_0})^2 - 2\frac{r}{x_0}\cos\theta\right]^{-1/2} \quad (4)$$

$$\frac{1}{R} = \frac{1}{x_0}\left[1 + \frac{r}{x_0} + (\frac{r}{x_0})^2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) + \ldots..\right] \quad (5)$$

$$\frac{1}{R} = \frac{1}{x_0}\sum_{l=0}^{\infty}\left(\frac{r}{x_0}\right)^l P_l(\cos\theta) \quad (6)$$

where $P_l(\cos\theta)$ is the Legendre polynomial of order l. Because Legendre polynomials have magnitudes less than unity for $l > 0$, and the infinite series in the above equation which is bounded and converges as $l \to \infty$.

The spherical harmonic in fractional space for the case $r < x_0$;

$$\frac{1}{R} = \frac{1}{x_0}\sum_{l=0}^{\infty}\left(\frac{r}{x_0}\right)^l P_l^{\alpha/2-1}(\cos\theta) \quad (7)$$

A similar approach for $r > x_0$ can be expressed as follows:

$$\frac{1}{R} = \frac{1}{r_0}\left[1 + (\frac{x_0}{r})^2 - 2\frac{x_0}{r}\cos\theta\right]^{-1/2} \quad (8)$$

$$\frac{1}{R} = \frac{1}{r}\left[1 + \frac{x_0}{r} + (\frac{x_0}{r})^2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) + \ldots..\right] \quad (9)$$

$$\frac{1}{R} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{x_0}{r}\right)^l P_l(\cos\theta) \quad (10)$$

The spherical harmonic in fractional space for the case $r > x_0$;

$$\frac{1}{R} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{x_0}{r}\right)^l P_l^{\alpha/2-1}(\cos\theta) \quad (11)$$

Similarly, the infinite series in the above equation which is also bounded and converges as $l \to \infty$. Therefore, the electric scalar potential Ψ can be expressed as an infinite sum of spherical harmonic modes, where:

$$\Psi = \frac{\rho I}{4\pi}\sum_{l=0}^{\infty}\frac{x_0^l P_l^{\alpha/2-1}(\cos\theta)}{r^{l+1}} \quad for\ x_0 < r \quad (12)$$

and

$$\Psi = \frac{\rho I}{4\pi}\sum_{l=0}^{\infty}\frac{r^l P_l^{\alpha/2-1}(\cos\theta)}{x_0^{l+1}} \quad for\ r < x_0 \quad (13)$$

2.1 Electric Potential for a Conducting Sphere in a Fractional Space

Let us now consider the electrical scalar potential at P in the presence of a conducting sphere of radius a and resistivity ρ_1, centred at the origin. Once again, a current of I is injected at $(x_0, 0,0)$. Due to the radial symmetry of the problem, $\partial/\partial\phi = 0$. Away from the source, the electric field is divergence free.

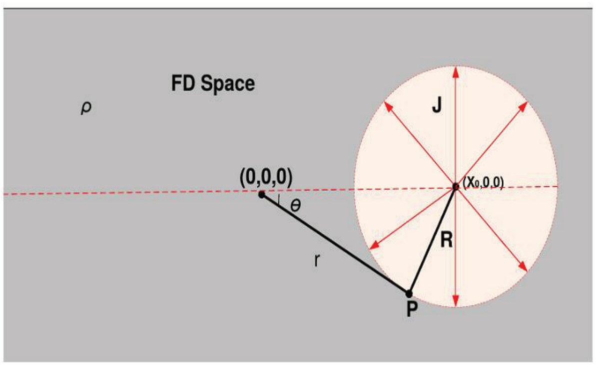

Fig. 1. The potential due to a point current source in fractional dimensional space

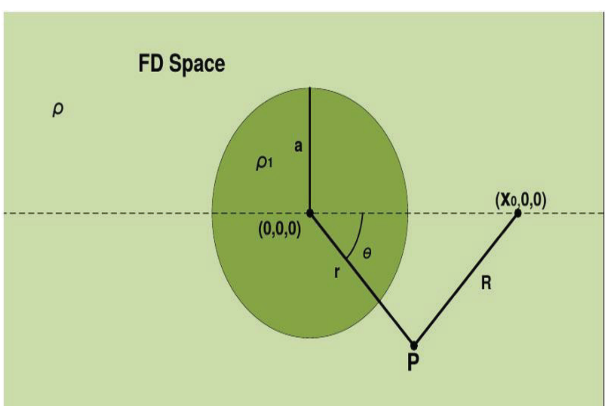

Fig. 2. The potential due to a conductive sphere embedded in fractional dimensional space

To solve this problem for fractional space we use the solution of laplacian equation in fractional space [14] and [15]:

$$\nabla^2\Psi(r,\theta) = \left(\frac{\partial^2}{\partial r^2} + \frac{\alpha-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\sin^{\alpha-2}\theta}\frac{\partial}{\partial\theta}\sin^{\alpha-2}\theta\frac{\partial}{\partial\theta}\right)\Psi(r,\theta) = 0 \quad (14)$$

Eq(3) is separable and suppose

$$\Psi(r,\theta) = R(r)\Theta(\theta) \quad (15)$$

The obtained angular and radial differential equations [16] are

$$\left[\frac{d^2}{d\theta^2} + (\alpha-2)\cot\theta\frac{d}{d\theta} + l(l+\alpha-2)\right]\Theta(\theta) = 0 \quad (16)$$

$$\left[\frac{d^2}{dr^2} + \frac{\alpha-1}{r}\frac{d}{dr} + \frac{l(l+\alpha-2)}{r^2}\right]R(r) = 0 \quad (17)$$

The solutions of the angular equation (5) are Gegenbauer polynomials in $cos\theta$ as explained in [14], namely

$$\Theta(\theta) = P_l^{\alpha/2-1}(\cos\theta),\ l = 0,1,2.. \quad (18)$$

From Eq(6), the radial differential equation gives the first few solutions such as

$$R_1(r) = r^l \quad (19)$$

$$R_2(r) = \frac{2}{r^{l+\alpha-2}} \quad (20)$$

Therefore, the general solutions $\Psi(r,\theta)$ in $\alpha-$ dimensional fractional space have the form

$$\Psi(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+\alpha-2}}\right)P_l^{\alpha/2-1}(\cos\theta) \quad (21)$$

where $P_1^{\alpha/2-1}(\cos\theta) = (\alpha-2)\cos\theta$, A_l and B_l are coefficients, which can be determined from the boundary conditions on $\Psi(r,\theta)$. Outside the potential should be bounded at infinity, so the positive power of r should not appear. Inside the sphere, the solution must be finite at origin, so the negative power of r must not appear inside the sphere. Hence the solution reduces to

$$\Psi_e^a(r,\theta) = \sum_{l=0}^{\infty}\frac{B_l}{r^{l+\alpha-2}}P_l^{\alpha/2-1}(\cos\theta),\ r > a \quad (22)$$

$$\Psi_i^a(r,\theta) = \sum_{l=0}^{\infty}A_l r^l P_l^{\alpha/2-1}(\cos\theta),\ r < a \quad (23)$$

These are the anomalous potentials. The physical requirement that the solution must be finite at the origin at infinity demands $B_l = 0$ inside the sphere and $A_l = 0$ out side the sphere. First, we take the source outside the sphere $z_0 > a$, so the total external potential $r > a$ can be expressed as

$$\Psi_e^a(r,\theta) = \frac{I\rho}{4\pi R} + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+\alpha-2}} P_l^{\alpha/2-1}(\cos\theta),\ r > a \quad (24)$$

using the value of $\frac{1}{R}$, we have the total potential outside the sphere is

$$\Psi_e^a(r,\theta) = \frac{I\rho}{4\pi}\frac{1}{x_0}\sum_{l=0}^{\infty}\left[(\frac{r}{z_0})^l P_l^{\alpha/2-1}(\cos\theta)\right] + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+\alpha-2}} P_l^{\alpha/2-1}(\cos\theta),\ r > a \quad (25)$$

To find the unknown coefficients, we impose the boundary conditions

$$\Psi_i^a = \Psi_e^a,\ \rho_1\frac{\partial\Psi_i^a}{\partial r} = \rho\frac{\partial\Psi_i^a}{\partial r},\ r = a \quad (26)$$

As a limiting case $a/x_0 << 1$, then the external potential can be expressed as

$$\Psi_e^a(r,\theta) \approx \frac{I\rho}{4\pi R} + \frac{B_1}{r^{\alpha-1}}(\alpha-2)\cos\theta,\ r > a \quad (27)$$

where

$$B_1 = \frac{I\rho}{4\pi}\frac{a^\alpha}{z_0^2}\frac{\rho_1-\rho}{\rho+(\alpha-1)\rho_1} \quad (28)$$

The potential can also be written as

$$\Psi_e^a(r,\theta) \approx \frac{I\rho}{4\pi R} + \frac{\rho-\rho_1}{\rho+(\alpha-1)\rho_1}a^\alpha E_0\frac{(\alpha-2)\cos\theta}{r^{\alpha-1}},\ r > a \quad (29)$$

Here, $E_0 = -\frac{I\rho}{4\pi z_0^2}$ is the primary electric field in the $\theta = 0$ direction that would exist at the origin in the absence of the spherical target. Assume the primary field is uniform, so it is taken the primary potential as $-E_0 r(\alpha-2)\cos\theta$. The complete solution for the external potential is

$$\Psi_e^a(r,\theta) = \frac{I\rho}{4\pi}\left[\frac{1}{R} + \sum_{l=0}^{\infty}\frac{a^\alpha}{(x_0 r)^{\alpha-1}}\frac{(\alpha-2)(\rho-\rho_1)}{(\alpha-2)\rho+(\alpha-1)\rho_1}E_0\frac{(\alpha-2)\cos\theta}{r^{\alpha-1}}\right],\ r > a \quad (30)$$

In geophysical prospecting, it is necessary to consider the anomalous field only and to measure the gradient of the potential such that

$$E = -\nabla\Psi_e^a(r,\theta) = \frac{\rho-\rho_1}{\rho+(\alpha-1)\rho_1}E_0R^3(\alpha-2)/(x_0^{\alpha-1})\left[\frac{(\alpha-1)x^2-y^2-z^2}{r^{\alpha+2}}i + \frac{\alpha xy}{r^{\alpha+2}}j + \frac{\alpha xz}{r^{\alpha+2}}k\right] \quad (31)$$

where $r = (x^2+y^2+z^2)^{1/2}$ and $x = r\cos\theta$.

where

$$B_l = \frac{I\rho}{4\pi}\frac{a^\alpha}{(x_0 r)^{\alpha-1}}\frac{\rho_1-\rho)}{\rho+(\alpha-1)\rho_1} \quad (32)$$

and

$$A_l = \frac{I\rho}{4\pi}\frac{1}{(x_0)^{\alpha-1}}\frac{\rho_1}{\rho+(\alpha-1)\rho_1} \quad (33)$$

Usually, we confine to the measure of the applied field, i.e. the x-direction, we require only the quantity

$$p = \frac{\rho-\rho_1}{\rho+(\alpha-1)\rho_1}E_0R^3(\alpha-2)/(x_0)^{\alpha-1} \quad (34)$$

This is the induced electric dipole moment of the sphere. We find it a useful result [16] as follow;

$$\Psi_e^a(r,\theta) \approx \frac{I\rho}{4\pi R} + \frac{CM}{4\pi r^{\alpha-1}}(\alpha-2)\cos\theta,\ r > a \quad (35)$$

where $CM = \frac{\rho-\rho_1}{\rho+(\alpha-1)\rho_1}\frac{4\pi a^\alpha}{\rho}E_0$

This is the induced current moment that rises to the secondary field.

For $\alpha = 3$, we retrieve the solution of the published problem [16] for integer order.

$$\Psi_e^a(r,\theta) = \frac{I\rho}{4\pi}\left[\frac{1}{R} + \frac{R^3}{(x_0 r)^2}\frac{\rho-\rho_1}{\rho+2\rho_1}E_0\frac{\cos\theta}{r^2}\right],\ r > a \quad (36)$$

$$E = \frac{\rho-\rho_1}{\rho+2\rho_1}E_0R^3/(x_0^2)\left[\frac{2x^2-y^2-z^2}{r^5}i + \frac{3xy}{r^5}j + \frac{3xz}{r^5}k\right] \quad (37)$$

where

$$B_1 = \frac{I\rho}{4\pi}\frac{a^3}{(x_0 r)^2}\frac{(\rho_1-\rho}{\rho+2\rho_1} \quad (38)$$

and

$$A_1 = \frac{I\rho}{4\pi}\frac{1}{(x_0)^2}\frac{\rho_1}{\rho+2\rho_1} \quad (39)$$

We use the same solution for $l = n = 0,1,2,3,\ldots..\mathrm{inf}$ to retrieve the published problem for azimuthal symmetry [17] and [18].

$$\Psi_e^a(r,\theta) = \frac{I\rho}{4\pi}\left[\frac{1}{R} + \sum_{l=0}^{\infty}\frac{a^{2n+1}}{(x_0 r)^{n+1}}\frac{(n)(\rho-\rho_1)}{n(\rho+(n+1)\rho_1)}E_0\frac{(n)\cos\theta}{r^{n+1}}\right],\ r > a \quad (40)$$

and

$$\Psi_i^{a}(r,\theta) = \frac{I\rho}{4\pi}\left[\frac{1}{R} + \sum_{l=0}^{\infty} \frac{1}{(x_0)^{n+1}} \frac{(2n+)\rho_1}{n(\rho+(n+1)\rho_1)} E_0 \frac{n\cos\theta}{r^{n+1}}\right],\ r < a \quad (41)$$

2.2 Special Case

The solution for the conductive sphere reduces to the perfect conducting sphere [16] if $\rho_1 \to 0$.

$$\Psi = \frac{I\rho}{4\pi R} + \frac{\hat{I}\rho}{4\pi\hat{R}},\ r > a,\ x_0 > a \quad (42)$$

$$\frac{1}{\hat{R}} = \sum_{l=0}^{\infty}\left[\left(\frac{(x_0)^{l+1}}{r^l} P_l^{\alpha/2-1}(\cos\theta)\right]\right. \quad (43)$$

$$\hat{R} = (r^2 + (x_0)^2 - 2rx_0\cos\theta)^{1/2} \quad (44)$$

where $\hat{I} = -(a/x_0)I$ and $\hat{x}_0 = a^2/x_0$

3. CONCLUSIONS

We have concluded that fractional dynamics plays an important role in describing the complex phenomena. In this paper, one of the fundamental equations in electromagnetism is the Laplace equation that has been solved in fractional α-dimensional space to find the apparent resistivity and electric field intensity. Due to the conductive sphere and point current source, the potential is obtained in fractional space. The mathematical problem simply describes and highlights many remote probing situations. That deals with the variation in the apparent resistivity using single- and cross-borehole probing. For all the calculated results, the classical results are recovered corresponding to the non-integer dimensional space parameter α=3.

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