Proceedings of the Pakistan Academy of Sciences:
A. Physical and Computational Sciences 53 (1): 1–16 (2016)

ISSN: 0377 - 2969 (print), 2306 - 1448 (online)

Pakistan Academy of Sciences

Research Article

Some Properties of Certain New Subclasses of Analytic Functions

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Abstract: In this paper, we introduce some new subclasses of analytic functions related to starlike, convex, close-to-convex and quasi-convex functions defined by using a generalized operator. Inclusion relationships for these subclasses are established. Also, we introduce some integral-preserving properties. Moreover, connections of the results presented here with those obtained in earlier works are pointed out.

Keywords: Analytic function, starlike function, convex function, close-to-convex, quasi-convex,
2010 MSC: 30C45

1. INTRODUCTION

Let $\mathbf{A}$ denotes the class of functions $f(z)$ which are analytic in $U=\{z\in\mathbb{C}:|z|<1\}$ and be given by

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n. \tag{1}$$

Let S denotes the subclass of $\mathbf{A}$ consisting of univalent functions in U. A function $f(z)\in S$ is called starlike function of order $\alpha(0\le\alpha<1)$ if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha\left(0\le\alpha<1; z\in U\right), \tag{2}$$

we denote the class of starlike functions by $S^*(\alpha)$. Also, a function $f(z)\in S$ is called convex function of order $\alpha(0\le\alpha<1)$ if and only if

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha\left(0\le\alpha<1; z\in U\right). \tag{3}$$

We denote this class by $C(\alpha)$. It is well known that:

$$f(z)\in C(\alpha)\Leftrightarrow zf'(z)\in S^*(\alpha)\left(0\le\alpha<1; z\in U\right). \tag{4}$$

A function $f(z)\in\mathbf{A}$ is called close-to-convex function of order $\beta(0\le\beta<1)$ and type $\alpha(0\le\alpha<1)$, if there exist a function $g(z)\in S^*(\alpha)$ such that

Received, January 2015; Accepted, January 2016
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$$Re\left\{\frac{zf'(z)}{g(z)}\right\}>\beta\left(0\le\beta<1;z\in U\right), \tag{5}$$

we denote this class by $K(\beta,\alpha)$. Also, a function $f(z)\in \mathbf{A}$ is called quasi-convex function of order $\beta(0\le\beta<1)$ and type $\alpha(0\le\alpha<1)$, if there exists a function $g(z)\in C(\alpha)$ such that

$$Re\left\{\frac{\left(zf'(z)\right)'}{g'(z)}\right\}>\beta\left(0\le\beta<1;z\in U\right), \tag{6}$$

we denote this class by $K^*(\beta,\alpha)$. Similarly, It is well known that:

$$f(z)\in K^*(\beta,\alpha)\Leftrightarrow zf'(z)\in K(\beta,\alpha)\ \left(0\le\alpha,\beta<1;z\in U\right). \tag{7}$$

For further information about starlike, convex, close-to-convex and quasi-convex function, see [18], [23], [25], [27], [31], [33], [36] and [37] etc.

Following the recent work of El-Ashwah and Aouf [11] and [10, with $p=1$], for $m\in\mathbb{N}_0=\{0,1,2,...\}$, $\lambda>0$, $l>-1$ and $f(z)\in\mathbf{A}$ given by (1), the integral operator $L_{\lambda,l}^m$ is defined as follows:

$$L_{\lambda,l}^m f(z)=\begin{cases} f(z), & m=0,\\ \frac{l+1}{\lambda}z^{1-\frac{l+1}{\lambda}}\int_0^z t^{\frac{l+1}{\lambda}-2}L_{\lambda,l}^{m-1}f(t)dt, & m=1,2,... .\end{cases} \tag{8}$$

It is clear from (8) that:

$$L_{\lambda,l}^m f(z)=z+\sum_{n=2}^{\infty}\left(\frac{l+1}{l+\lambda(n-1)+1}\right)^m a_n z^n\ \ (\lambda>0;l>-1;m\in\mathbb{N}_0). \tag{9}$$

Also, for $A>0$ and $a,c\in\mathbb{C}$, are such that $Re\{c-a\}\ge 0$, Raina and Sharma [32] defined the integral operator $J_A^{a,c}:\ \mathbf{A}\to\mathbf{A}$, as follows:

(i) for $Re\{c-a\}>0$ and $Re\{a\}>-A$ by

$$J_A^{a,c}f(z)=\frac{\Gamma(c+A)}{\Gamma(a+A)}\frac{1}{\Gamma(c-a)}\int_0^1(1-t)^{c-a-1}t^{a-1}f(zt^A)dt; \tag{10}$$

(ii) for $a=c$ by

$$J_A^{a,a}f(z)=f(z). \tag{11}$$

For $f(z)$ defined by (1), it is easily from (10) and (11) that:

$$J_A^{a,c}f(z)=z+\frac{\Gamma(c+A)}{\Gamma(a+A)}\sum_{n=2}^{\infty}\frac{\Gamma(a+nA)}{\Gamma(c+nA)}a_n z^n. \tag{12}$$

$$(A>0;a,c\in\mathbb{C};Re\{c-a\}\ge 0;Re\{a\}>-A)$$

By combining the two linear operators $L_{\lambda,l}^m$ and $J_A^{a,c}$. Then, the generalized operator

$$I_{\lambda,l}^m(a,c,A):\ \mathbf{A}\to\mathbf{A},$$

is defined for the purpose of this paper as follows:

$$I_{\lambda,l}^m(a,c,A)f(z)=L_{\lambda,l}^m\left(J_A^{a,c}f(z)\right)=J_A^{a,c}\left(L_{\lambda,l}^m f(z)\right), \tag{13}$$

which can be easily expressed as follows:

$$I_{\lambda,l}^{m}(a,c,A)f(z) = z + \frac{\Gamma(c+A)}{\Gamma(a+A)} \sum_{n=2}^{\infty} \frac{\Gamma(a+nA)}{\Gamma(c+nA)} \left(\frac{l+1}{l+\lambda(n-1)+1} \right)^{m} a_n z^n, \tag{14}$$

$$(A > 0; a, c \in \mathbb{C}; Re\{c-a\} \geq 0; Re\{a\} > -A; \lambda > 0; l > -1; m \in \mathbb{N}_0).$$

In view of (1.8), (1.11) and (1.13), it is clear that:

$$I_{\lambda,l}^{0}(a,c,A)f(z) = J_{A}^{a,c} f(z) \quad \text{and} \quad I_{\lambda,l}^{m}(a,a,A)f(z) = L_{\lambda,l}^{m} f(z). \tag{15}$$

By specializing the parameters in (1.14), we note that the operator $I_{\lambda,l}^{m}(a,c,A)$ generalizes a lot of previous operators, as follows:

(i) $I_{\lambda,l}^{m}(\mu-1,0,1)f(z) = I_{\lambda,l,\mu}^{m} f(z) \left(\lambda > 0; l > -1; \mu > 0; m \in \mathbb{N}_0 = \{0,1,2,...\}\right)$ (see Aouf and El-Ashwah [3]);

(ii) $I_{1,l}^{s}(\mu-1,0,1)f(z) = I_{l,\mu}^{s} f(z) \left(l > -1; \mu > 0; s \in \mathbb{R}\right)$ (see Cho and Kim [8]);

(iii) $I_{\lambda,0}^{m}(\mu-1,0,1)f(z) = I_{\lambda,\mu}^{m} f(z) \left(\lambda > 0; \mu > 0; m \in \mathbb{Z}\right)$ (see Aouf et al. [2]);

(iv) $I_{\lambda,l}^{-n}(a,a,A)f(z) = I^{n}(\lambda,l)f(z) \left(\lambda > 0; l > -1; n \in \mathbb{N}_0\right)$ (see Catas [7]);

(v) $I_{\lambda,l}^{m}(a,a,A)f(z) = J^{m}(\lambda,l)f(z) \left(\lambda > 0; l > -1; m \in \mathbb{N}_0\right)$ (see El-Ashwah and Aouf [10]);

(vi) $I_{\lambda,0}^{-n}(a,a,A)f(z) = I_{\lambda}^{n} f(z) \left(\lambda > 0; n \in \mathbb{Z}\right)$ (see Patel [30]);

(vii) $I_{1,\alpha-1}^{\mu}(a,a,A)f(z) = L_{\alpha}^{\mu} f(z) \left(\mu > 0; \alpha > 0\right)$ (see Komatu [16]; Aouf [1]);

(viii) $I_{1,1}^{\sigma}(a,a,A)f(z) = L^{\sigma} f(z) \left(\sigma > 0\right)$ (see Jung et al. [15]; Liu [19]);

(ix) $I_{1,1}^{\beta}(a,a,A)f(z) = L^{\beta} f(z) \left(\beta \in \mathbb{Z}\right)$ (see Uralegaddi and Somanatha [39]; Flett [12]);

(x) $I_{1,0}^{n}(a,a,A)f(z) = I^{n} f(z) \left(n \in \mathbb{N}_0\right)$ and $I_{1,0}^{-n}(a,a,A)f(z) = D^{n} f(z) \left(n \in \mathbb{N}_0\right)$ (see Salagean [35]);

(xi) $I_{1,l}^{\mu}(a,a,A)f(z) = P_{l}^{\mu} f(z) \left(\mu > 0; l > -1\right)$ (see Gao et al. [13]);

(xii) $I_{1,\sigma}^{1}(a,a,A)f(z) = L_{\sigma} f(z) \left(\sigma > 0\right)$ (see Owa and Srivastava [29]; Srivastava and Owa [38]);

(xiii) $I_{\lambda,l}^{0}(\beta,\alpha+\beta-\gamma+1,1)f(z) = \Re_{\beta}^{\alpha,\gamma} f(z) \left(\gamma > 0; \alpha \geq \gamma-1; \beta > -1\right)$ (see Aouf et al. [4]);

(xiv) $I_{\lambda,l}^{0}(\beta,\alpha+\beta,1)f(z) = Q_{\beta}^{\alpha} f(z) \left(\alpha \geq 0; \beta > -1\right)$ (see Liu and Owa [21]; Jung et al. [15]; Li [17]; Liu [20]);

(xv) $I_{\lambda,l}^{0}(a-1,c-1,1)f(z) = L(a,c)f(z) \left(a,c \in \mathbb{C} \setminus \mathbb{Z}_0^-, \mathbb{Z}_0^- = \{0,-1,-2,...\}\right)$ (see Carlson and Shaffer [6]);

(xvi) $I_{\lambda,l}^{0}(\mu-1,\nu,1)f(z) = I_{\nu,\mu} f(z) \left(\mu > 0; \nu > -1\right)$ (see Choi et al. [9]);

(xvii) $I_{\lambda,l}^{0}(\alpha,0,1)f(z) = D^{\alpha} f(z) \left(\alpha > -1\right)$ (see Ruscheweyh [34]);

(xviii) $I_{\lambda,l}^{0}(1,n,1)f(z) = D^{n} f(z) \left(n \in \mathbb{N}_0\right)$ (see Noor [26]; Noor and Noor [28]).

Using (1.14), we can obtain the following recurrence relations, which are needed for our proofs in following two sections:

$$z\left(I_{\lambda,l}^{m+1}(a,c,A)f(z)\right)' = \frac{1+l}{\lambda} I_{\lambda,l}^{m}(a,c,A)f(z) - \frac{1+l-\lambda}{\lambda} I_{\lambda,l}^{m+1}(a,c,A)f(z), \tag{16}$$

$$z\left(I_{\lambda,l}^{m}(a,c,A)f(z)\right)' = \frac{a+A}{A} I_{\lambda,l}^{m}(a+1,c,A)f(z) - \frac{a}{A} I_{\lambda,l}^{m}(a,c,A)f(z). \qquad (17)$$

Definition 1. By using the operator $I_{\lambda,l}^{m}(a,c,A)f(z)$ defined by (14), we introduce the following subclasses of the class $\mathbf{A}$, as follows:

$$S_{\lambda,l}^{*m}(a,c,A;\alpha) = \left\{f : f(z) \in \mathbf{A} \text{ and } I_{\lambda,l}^{m}(a,c,A)f(z) \in S^{*}(\alpha)\right\}, \qquad (18)$$

$$C_{\lambda,l}^{m}(a,c,A;\alpha) = \left\{f : f(z) \in \mathbf{A} \text{ and } I_{\lambda,l}^{m}(a,c,A)f(z) \in C(\alpha)\right\}, \qquad (19)$$

$$K_{\lambda,l}^{m}(a,c,A;\beta,\alpha) = \left\{f : f(z) \in \mathbf{A} \text{ and } I_{\lambda,l}^{m}(a,c,A)f(z) \in K(\beta,\alpha)\right\}, \qquad (20)$$

$$K_{\lambda,l}^{*m}(a,c,A;\beta,\alpha) = \left\{f : f(z) \in \mathbf{A} \text{ and } I_{\lambda,l}^{m}(a,c,A)f(z) \in K^{*}(\beta,\alpha)\right\}, \qquad (21)$$

$$(A > 0, a,c \in \mathbb{C}, Re\left\{c-a\right\} \geq 0, Re\left\{a\right\} > -A, \lambda > 0, l > -1, 0 \leq \alpha,\beta < 1, m \in \mathbb{N}_0).$$

Remark 1. If we set $a = c$ in Definition 1, we obtain the following subclasses of $\mathbf{A}$:

$$S_{\lambda,l}^{*m}(\alpha) = \left\{f : f(z) \in \mathbf{A} \text{ and } L_{\lambda,l}^{m}f(z) \in S^{*}(\alpha)\right\}, \qquad (22)$$

$$C_{\lambda,l}^{m}(\alpha) = \left\{f : f(z) \in \mathbf{A} \text{ and } L_{\lambda,l}^{m}f(z) \in C(\alpha)\right\}, \qquad (23)$$

$$K_{\lambda,l}^{m}(\beta,\alpha) = \left\{f : f(z) \in \mathbf{A} \text{ and } L_{\lambda,l}^{m}f(z) \in K(\beta,\alpha)\right\}, \qquad (24)$$

$$K_{\lambda,l}^{*m}(\beta,\alpha) = \left\{f : f(z) \in \mathbf{A} \text{ and } L_{\lambda,l}^{m}f(z) \in K^{*}(\beta,\alpha)\right\}. \qquad (25)$$

Where $L_{\lambda,l}^{m}f(z)$ is defined by (9).

Remark 2. If we set $m = 0$ in Definition 1, we obtain the following subclasses of $\mathbf{A}$:

$$S^{*}(a,c,A;\alpha) = \left\{f : f(z) \in \mathbf{A} \text{ and } J_{A}^{a,c}f(z) \in S^{*}(\alpha)\right\}, \qquad (26)$$

$$C(a,c,A;\alpha) = \left\{f : f(z) \in \mathbf{A} \text{ and } J_{A}^{a,c}f(z) \in C(\alpha)\right\}, \qquad (27)$$

$$K(a,c,A;\beta,\alpha) = \left\{f : f(z) \in \mathbf{A} \text{ and } J_{A}^{a,c}f(z) \in K(\beta,\alpha)\right\}, \qquad (28)$$

$$K^{*}(a,c,A;\beta,\alpha) = \left\{f : f(z) \in \mathbf{A} \text{ and } J_{A}^{a,c}f(z) \in K^{*}(\beta,\alpha)\right\}. \qquad (29)$$

Where $J_{A}^{a,c}f(z)$ is defined by (12).

In order to introduce our main results, we shall need the following lemma which is given by Miller and Mocanu [24].

Lemma 1. *Let* $u = u_1 + iu_2$, $v = v_1 + iv_2$ *and let* $\psi(u,v)$ *be a complex-valued function such that*

$$\psi : D \to \mathbb{C} \quad (D \subset \mathbb{C}\times\mathbb{C}).$$

Suppose also that the function $\psi(u,v)$ satisfies each of the following conditions:

(i) $\psi(u,v)$ *is continuous in* D;

(ii) $(1,0) \in D$ *and* $Re\left\{\psi(1,0)\right\} > 0$;

(iii) $Re\left\{\psi(iu_2,v_1)\right\} \leq 0$ *for all* $(iu_2,v_1) \in D$ *such that*

$$v_1 \leq -\frac{1}{2}(1+u_2^2). \qquad (30)$$

Let

$$h(z)=1+c_1 z+c_2 z^2+\dots, \tag{31}$$

be analytic in U *such that* $(h(z),zh'(z))\in D$ $(z\in U)$. *If* $Re\{\psi(h(z),zh'(z))\}>0$ $(z\in U)$, *then* $Re\{h(z)\}>0$ *for* $z\in U$.

2. INCLUSION RELATIONSHIPS

Unless otherwise mentioned, we shall assume throughout the paper that $A>0$, $a,c\in\mathbb{C}$, $Re\{c-a\}\geq 0$, $Re\{a\}>-A$, $\lambda>0$, $l>-1$, $0\leq\alpha,\beta<1$, $m\in\mathbb{N}_0$ and $f(z)\in\mathbf{A}$.

In this section, we give several inclusion relationships for analytic function classes, which are associated with the generalized operator $I_{\lambda,l}^{m}(a,c,A)f(z)$ defined by (14).

Theorem 1. *Let* $a=a_1+ia_2$ with $Re\{a\}>-A\alpha$, *then*

$$S_{\lambda,l}^{*m}(a+1,c,A;\alpha)\subset S_{\lambda,l}^{*m}(a,c,A;\alpha)\subset S_{\lambda,l}^{*m+1}(a,c,A;\alpha). \tag{32}$$

Proof**.** (i) we begin with showing the first inclusion relationship

$$S_{\lambda,l}^{*m}(a+1,c,A;\alpha)\subset S_{\lambda,l}^{*m}(a,c,A;\alpha), \tag{33}$$

which is asserted by Theorem 1. Let $f(z)\in S_{\lambda,l}^{*m}(a+1,c,A;\alpha)$ and set

$$\frac{z\left(I_{\lambda,l}^{m}(a,c,A)f(z)\right)'}{I_{\lambda,l}^{m}(a,c,A)f(z)}-\alpha=(1-\alpha)h(z), \tag{34}$$

where $h(z)$ is defined by (31). By using the identity (17) and (34), we obtain

$$\frac{a+A}{A}\frac{I_{\lambda,l}^{m}(a+1,c,A)f(z)}{I_{\lambda,l}^{m}(a,c,A)f(z)}=(1-\alpha)h(z)+\alpha+\frac{A}{a}. \tag{35}$$

By using logarithmic differentiation on both sides of (35), we obtain

$$\frac{z\left(I_{\lambda,l}^{m}(a+1,c,A)f(z)\right)'}{I_{\lambda,l}^{m}(a+1,c,A)f(z)}=\frac{z\left(I_{\lambda,l}^{m}(a,c,A)f(z)\right)'}{I_{\lambda,l}^{m}(a,c,A)f(z)}+\frac{(1-\alpha)zh'(z)}{(1-\alpha)h(z)+\alpha+\frac{a}{A}},$$

using (34) in the above equation, we obtain

$$\frac{z\left(I_{\lambda,l}^{m}(a+1,c,A)f(z)\right)'}{I_{\lambda,l}^{m}(a+1,c,A)f(z)}-\alpha=(1-\alpha)h(z)+\frac{(1-\alpha)zh'(z)}{(1-\alpha)h(z)+\alpha+\frac{a}{A}}.$$

or

$$\frac{z\left(I_{\lambda,l}^{m}(a+1,c,A)f(z)\right)'}{I_{\lambda,l}^{m}(a+1,c,A)f(z)}-\alpha=(1-\alpha)h(z)+\frac{A(1-\alpha)zh'(z)}{A\left[(1-\alpha)h(z)+\alpha\right]+a}. \tag{36}$$

Now, we choose $u=h(z)=u_1+iu_2$ and $v=zh'(z)=v_1+iv_2$, and define the function $\psi(u,v)$ by

$$\psi(u,v)=(1-\alpha)u+\frac{A(1-\alpha)v}{A\left[(1-\alpha)u+\alpha\right]+a}.$$

It is easy to see that the function $\psi(u,v)$ satisfies conditions (i) and (ii) of Lemma 1 in $D=\left(\mathbb{C}\setminus\left\{\frac{\alpha A+a}{A(\alpha-1)}\right\}\right)\times\mathbb{C}.$ Also, we verify condition (iii) as follows:

$$\operatorname{Re}\{\psi(iu_2,v_1)\}=\operatorname{Re}\left\{\frac{A(1-\alpha)v_1}{A\left[(1-\alpha)iu_2+\alpha\right]+a}\right\}=\frac{A\left(\alpha A+a_1\right)(1-\alpha)v_1}{\left[\left(A\alpha+a_1\right)^2+\left(A(1-\alpha)u_2+a_2\right)^2\right]}$$

$$\leq-\frac{A\left(\alpha A+a_1\right)(1-\alpha)(1+u_2^2)}{2\left[\left(A\alpha+a_1\right)^2+\left(A(1-\alpha)u_2+a_2\right)^2\right]}<0.$$

Which shows that $\psi(u,v)=\psi(h(z),zh'(z))$ $(z\in U)$ satisfies the hypotheses of the Lemma 1, then $Re\{h(z)\}>0$ $(z\in U)$, then using (34), we have $f(z)\in S^{*m}_{\lambda,l}(a,c,A;\alpha).$ This completes the proof of (33).

(ii) Now, we prove

$$S^{*m}_{\lambda,l}(a,c,A;\alpha)\subset S^{*m+1}_{\lambda,l}(a,c,A;\alpha), \tag{37}$$

which is the second inclusion relationship of Theorem 1. Let $f(z)\in S^{*m}_{\lambda,l}(a,c,A;\alpha)$ and set

$$\frac{z\left(I^{m+1}_{\lambda,l}(a,c,A)f(z)\right)'}{I^{m+1}_{\lambda,l}(a,c,A)f(z)}-\alpha=(1-\alpha)h(z), \tag{38}$$

where $h(z)$ is defined by (31). By applying the identity (16) in (38), we obtain

$$\frac{1+l}{\lambda}\frac{I^{m}_{\lambda,l}(a,c,A)f(z)}{I^{m+1}_{\lambda,l}(a,c,A)f(z)}=(1-\alpha)h(z)+\alpha+\frac{1+l-\lambda}{\lambda}. \tag{39}$$

By using logarithmic differentiation on both sides of (39), we obtain

$$\frac{z\left(I^{m}_{\lambda,l}(a,c,A)f(z)\right)'}{I^{m}_{\lambda,l}(a,c,A)f(z)}-\frac{z\left(I^{m+1}_{\lambda,l}(a,c,A)f(z)\right)'}{I^{m+1}_{\lambda,l}(a,c,A)f(z)}=\frac{(1-\alpha)zh'(z)}{(1-\alpha)h(z)+\alpha+\frac{1+l-\lambda}{\lambda}},$$

using (38) in the above equation, we have

$$\frac{z\left(I^{m}_{\lambda,l}(a,c,A)f(z)\right)'}{I^{m}_{\lambda,l}(a,c,A)f(z)}-\alpha=(1-\alpha)h(z)+\frac{\lambda(1-\alpha)zh'(z)}{\lambda\left((1-\alpha)h(z)+\alpha\right)+\left(1+l-\lambda\right)}. \tag{40}$$

By using arguments similar to those detailed before, together with (40) and $\psi(u,v)$ is continuous in $D=\left(\mathbb{C}\setminus\left\{1-\frac{l+1}{\lambda(1-\alpha)}\right\}\right)\times\mathbb{C}$, then we can prove (37), which is the second inclusion relationship of Theorem 1. Combining the inclusion relationships (33) and (37), we complete the proof of Theorem 1.

Theorem 2. *Let* $a=a_1+ia_2$ with $Re\{a\}>-A\alpha$*, then*

$$C^{m}_{\lambda,l}(a+1,c,A;\alpha)\subset C^{m}_{\lambda,l}(a,c,A;\alpha)\subset C^{m+1}_{\lambda,l}(a,c,A;\alpha). \tag{41}$$

Proof. We first show that

$$C^{m}_{\lambda,l}(a+1,c,A;\alpha)\subset C^{m}_{\lambda,l}(a,c,A;\alpha). \tag{42}$$

Let $f(z)\in C^{m}_{\lambda,l}(a+1,c,A;\alpha)$. Then, using Theorem 1, we have

$$
\begin{aligned}
I_{\lambda,l}^{m}(a+1,c,A)f(z)\in C(\alpha) \quad &\Leftrightarrow z\left(I_{\lambda,l}^{m}(a+1,c,A)f(z)\right)'\in S^{*}(\alpha)\\
&\Leftrightarrow I_{\lambda,l}^{m}(a+1,c,A)zf'(z)\in S^{*}(\alpha)\\
&\Leftrightarrow zf'(z)\in S_{\lambda,l}^{*m}(a+1,c,A;\alpha)\\
&\Rightarrow zf'(z)\in S_{\lambda,l}^{*m}(a,c,A;\alpha)\\
&\Leftrightarrow I_{\lambda,l}^{m}(a,c,A)zf'(z)\in S^{*}(\alpha)\\
&\Leftrightarrow z\left(I_{\lambda,l}^{m}(a,c,A)f(z)\right)'\in S^{*}(\alpha)\\
&\Leftrightarrow I_{\lambda,l}^{m}(a,c,A)f(z)\in C(\alpha)\\
&\Leftrightarrow f(z)\in C_{\lambda,l}^{m}(a,c,A;\alpha).
\end{aligned}
$$

This completes the proof of (42). By using arguments similar to those detailed above, we can also prove the right part of Theorem 2, that is, that

$$C_{\lambda,l}^{m}(a,c,A;\alpha)\subset C_{\lambda,l}^{m+1}(a,c,A;\alpha). \tag{43}$$

Combining the inclusion relationships (42) and (43), then the proof of Theorem 2 is completed.

Theorem 3. *Let* $a=a_1+ia_2$ with $Re\{a\}>-A\alpha$, *then*

$$K_{\lambda,l}^{m}(a+1,c,A;\beta,\alpha)\subset K_{\lambda,l}^{m}(a,c,A;\beta,\alpha)\subset K_{\lambda,l}^{m+1}(a,c,A;\beta,\alpha). \tag{44}$$

Proof. Let us begin with proving that

$$K_{\lambda,l}^{m}(a+1,c,A;\beta,\alpha)\subset K_{\lambda,l}^{m}(a,c,A;\beta,\alpha). \tag{45}$$

Let $f(z)\in K_{\lambda,l}^{m}(a+1,c,A;\beta,\alpha)$. Then there exists a function $g(z)\in S^{*}(\alpha)$ such that

$$Re\left\{\frac{z\left(I_{\lambda,l}^{m}(a+1,c,A)f(z)\right)'}{g(z)}\right\}>\beta\ \left(z\in U\right).$$

We put $g(z)=I_{\lambda,l}^{m}(a+1,c,A)k(z)$, so that we have $k(z)\in S_{\lambda,l}^{*m}(a+1,c,A;\alpha)$ and

$$Re\left\{\frac{z\left(I_{\lambda,l}^{m}(a+1,c,A)f(z)\right)'}{I_{\lambda,l}^{m}(a+1,c,A)k(z)}\right\}>\beta\ \left(z\in U\right).$$

Next, we put

$$\frac{z\left(I_{\lambda,l}^{m}(a,c,A)f(z)\right)'}{I_{\lambda,l}^{m}(a,c,A)k(z)}=\left(1-\beta\right)h(z)+\beta, \tag{46}$$

where $h(z)$ is given by (31). Thus, by using the identity (17), we obtain

$$
\begin{aligned}
\frac{z\left(I_{\lambda,l}^{m}(a+1,c,A)f(z)\right)'}{I_{\lambda,l}^{m}(a+1,c,A)k(z)}&=\frac{I_{\lambda,l}^{m}(a+1,c,A)zf'(z)}{I_{\lambda,l}^{m}(a+1,c,A)k(z)}\\
&=\frac{\frac{A}{A+a}z\left(I_{\lambda,l}^{m}(a,c,A)zf'(z)\right)'+\frac{a}{A+a}I_{\lambda,l}^{m}(a,c,A)zf'(z)}{\frac{A}{A+a}z\left(I_{\lambda,l}^{m}(a,c,A)k(z)\right)'+\frac{a}{A+a}I_{\lambda,l}^{m}(a,c,A)k(z)}\\
&=\frac{A\,\frac{z\left(I_{\lambda,l}^{m}(a,c,A)zf'(z)\right)'}{I_{\lambda,l}^{m}(a,c,A)k(z)}+a\,\frac{I_{\lambda,l}^{m}(a,c,A)zf'(z)}{I_{\lambda,l}^{m}(a,c,A)k(z)}}{A\,\frac{z\left(I_{\lambda,l}^{m}(a,c,A)k(z)\right)'}{I_{\lambda,l}^{m}(a,c,A)k(z)}+a}.
\end{aligned}
\tag{47}
$$

Since $k(z)\in S^{*m}_{\lambda,l}(a+1,c,A;\alpha)$, by using Theorem 1, we can put

$$\frac{z\left(I^m_{\lambda,l}(a,c,A)k(z)\right)'}{I^m_{\lambda,l}(a,c,A)k(z)}=(1-\alpha)G(z)+\alpha, \tag{48}$$

where $G(z)=g_1(x,y)+ig_2(x,y)$ and $Re\{G(z)\}=g_1(x,y)>0\ (z\in U)$. Using (46) and (48) in (47), we have

$$\frac{z\left(I^m_{\lambda,l}(a+1,c,A)f(z)\right)'}{I^m_{\lambda,l}(a+1,c,A)k(z)}=\frac{A\,\dfrac{z\left(I^m_{\lambda,l}(a,c,A)zf'(z)\right)'}{I^m_{\lambda,l}(a,c,A)k(z)}+a\left[\beta+(1-\beta)h(z)\right]}{A\left[(1-\alpha)G(z)+\alpha\right]+a}. \tag{49}$$

Moreover, from (46), we can put

$$z\left(I^m_{\lambda,l}(a,c,A)f(z)\right)'=\left[(1-\beta)h(z)+\beta\right]I^m_{\lambda,l}(a,c,A)k(z). \tag{50}$$

Differentiating both sides of (50) with respect to z, and using (46) and (48), we obtain

$$\frac{z\left(I^m_{\lambda,l}(a,c,A)zf'(z)\right)'}{I^m_{\lambda,l}(a,c,A)k(z)}=(1-\beta)zh'(z)+\left[(1-\beta)h(z)+\beta\right]\left[(1-\alpha)G(z)+\alpha\right]. \tag{51}$$

By substituting (51) into (49), we obtain

$$\frac{z\left(I^m_{\lambda,l}(a+1,c,A)f(z)\right)'}{I^m_{\lambda,l}(a+1,c,A)k(z)}-\beta=(1-\beta)h(z)+\frac{A(1-\beta)zh'(z)}{A\left[(1-\alpha)G(z)+\alpha\right]+a}. \tag{52}$$

In (52), take $u=h(z)=u_1+iu_2$, $v=zh'(z)=v_1+iv_2$ and define the function $\psi(u,v)$ by

$$\psi(u,v)=(1-\beta)u+\frac{A(1-\beta)v}{A\left[(1-\alpha)G(z)+\alpha\right]+a}, \tag{53}$$

where $(u,v)\in D=\mathbb{C}\times\mathbb{C}$. Then it follows from (53) that:

(i) $\psi(u,v)$ is continuous in D;

(ii) $(1,0)\in D$ and $Re\{\psi(1,0)\}=1-\beta>0$;

(iii) for all $(iu_2,v_1)\in D$ such that $v_1\le-\frac{1}{2}\left(1+u_2^2\right)$, we have

$$\begin{aligned}Re\{\psi(iu_2,v_1)\}&=Re\left\{\frac{A(1-\beta)v_1}{A\left[(1-\alpha)G(z)+\alpha\right]+a}\right\}\\&=\frac{\left(A(1-\beta)v_1\right)\left(A\left[(1-\alpha)g_1(z)+\alpha\right]+a_1\right)}{\left(A\left[(1-\alpha)g_1(z)+\alpha\right]+a_1\right)^2+\left(A(1-\alpha)g_2(z)+a_2\right)^2}\\&=-\frac{\left(A(1-\beta)\left(1+u_2^2\right)\right)\left(A\left[(1-\alpha)g_1(z)+\alpha\right]+a_1\right)}{2\left[\left(A\left[(1-\alpha)g_1(z)+\alpha\right]+a_1\right)^2+\left(A(1-\alpha)g_2(z)+a_2\right)^2\right]}\\&<0,\end{aligned}$$

which shows that $\psi(u,v)$ satisfies the hypotheses of Lemma 1. Thus, in light of (46), we easily deduce the inclusion relationship (45).

The remainder of our proof of Theorem 3 would make use of the identity (16) in an analogous manner. Therefore, we choose to omit the details involved.

Theorem 4. *Let* $a = a_1 + ia_2$ with $Re\{a\} > -A\alpha$*, then*

$$K_{\lambda,l}^{*m}(a+1,c,A;\beta,\alpha) \subset K_{\lambda,l}^{*m}(a,c,A;\beta,\alpha) \subset K_{\lambda,l}^{*m+1}(a,c,A;\beta,\alpha). \tag{54}$$

Proof. Just, as we derived Theorem 2 as a consequence of Theorem 1 by using the equivalence (4). Similarly, we can prove Theorem 4 by using Theorem 3 in conjunction with the equivalence (7). Therefore, we choose to omit the details involved.

Remark 3.

(i) Taking $m = s\ (s \in \mathbb{R})$, $\lambda = 1$, $a = \mu - 1(\mu > 0)$, $c = 0$ and $A = 1$ in Theorems 1-3, we obtain the results obtained by Cho and Kim [8, Theorems 2.1-2.3 with $\varphi(z) = \psi(z) = \frac{1+z}{1-z}$];

(ii) Taking $a = \mu - 1(\mu > 0)$, $c = 0$ and $A = 1$ in Theorems 1-3, we obtain the results obtained by Aouf and El-Ashwah [3, Theorems 1-3 with $\varphi(z) = \psi(z) = \frac{1+z}{1-z}$];

(iii) Taking $l = 0$, $a = \mu - 1(\mu > 0)$, $c = 0$ and $A = 1$ in Theorems 1-3, we obtain the results obtained by Aouf et al. [2, Theorems 1-3 with $\varphi(z) = \psi(z) = \frac{1+z}{1-z}$].

Taking $a = c$ in Theorems 1-4, we obtain the following corollary.

Corollary 1. For the subclasses $S_{\lambda,l}^{*m}(\alpha)$, $C_{\lambda,l}^{m}(\alpha)$, $K_{\lambda,l}^{m}(\beta,\alpha)$ and $K_{\lambda,l}^{*m}(\beta,\alpha)$ defined in Remark 1, we have the following inclusion relations.

$$S_{\lambda,l}^{*m}(\alpha) \subset S_{\lambda,l}^{*m+1}(\alpha),$$
$$C_{\lambda,l}^{m}(\alpha) \subset C_{\lambda,l}^{m+1}(\alpha),$$
$$K_{\lambda,l}^{m}(\beta,\alpha) \subset K_{\lambda,l}^{m+1}(\beta,\alpha),$$
$$K_{\lambda,l}^{*m}(\beta,\alpha) \subset K_{\lambda,l}^{*m+1}(\beta,\alpha).$$

Remark 4.

(i) Taking $\lambda = 1$, $m = \mu(\mu > 0)$ and $l = a - 1(a > 0)$ in Corollary 1, we obtain the results obtained by Aouf [1, Theorems 1-4];

(ii) Taking $\lambda = l = 1$ and $m = \sigma(\sigma > 0)$ in Corollary 1, we obtain the results obtained by Liu [19, Theorems 1-4].

Taking $m = 0$ in Theorems 1-4, we obtain the following corollary.

Corollary 2. For the subclasses $S^*(a,c,A;\alpha)$, $C(a,c,A;\alpha)$, $K(a,c,A;\beta,\alpha)$ and $K^*(a,c,A;\beta,\alpha)$ defined in Remark 2, we have the following inclusion relations.

$$S^*(a+1,c,A;\alpha) \subset S^*(a,c,A;\alpha),$$
$$C(a+1,c,A;\alpha) \subset C(a,c,A;\alpha),$$
$$K(a+1,c,A;\beta,\alpha) \subset K(a,c,A;\beta,\alpha),$$
$$K^*(a+1,c,A;\beta,\alpha) \subset K^*(a,c,A;\beta,\alpha).$$

Remark 5. Taking $\alpha=\beta=0$, $a=\mu-1(\mu>0)$, $c=\lambda(\lambda>-1)$ and $A=1$ in the first three inclusion relationships of Corollary 2, we obtain the results obtained by Choi et al. [9, Theorems 1-3, only the first parts with $\varphi(z)=\psi(z)=\frac{1+z}{1-z}$].

3. INTEGRAL-PRESERVING PROPERTIES INVOLVING THE INTEGRAL OPERATOR L_σ

For $\sigma>-1$ and $f(z)\in \mathbf{A}$, we recall the generalized Bernardi-Libera-Livingston integral operator $L_\sigma : \mathbf{A}\to\mathbf{A}$, as following (see Owa and H. M. Srivastava [29]):

$$L_\sigma f(z)=\frac{\sigma+1}{z^\sigma}\int_0^z t^{\sigma-1}f(t)dt$$

$$=z+\sum_{n=2}^{\infty}\left(\frac{\sigma+1}{\sigma+n}\right)a_n z^n \quad (\sigma>-1, f(z)\in\mathbf{A}). \tag{55}$$

The operator $L_\sigma f(z)(\sigma\in\mathbb{N})$ was introduced by Bernardi [5]. In particular, the operator $L_1 f(z)$ was studied earlier by Libera [18] and Livingston [22]. Using (14) and (55), it is clear that $L_\sigma f(z)$ satisfies the following relationship:

$$z\left(I_{\lambda,l}^m(a,c,A)L_\sigma f(z)\right)'=(\sigma+1)I_{\lambda,l}^m(a,c,A)f(z)-\sigma I_{\lambda,l}^m(a,c,A)L_\sigma f(z). \tag{56}$$

In order to obtain the integral-preserving properties involving the integral operator L_σ, we need the following lemma which is known as Jack's Lemma (see [14]).

Lemma 2. [14]**.** *Let $\omega(z)$ be a non-constant function analytic in U with $\omega(0)=0$. If $|\omega(z)|$ attains its maximum value on the circle $|z|=r<1$ at z_0, then*

$$z_0\omega'(z_0)=\zeta\omega(z_0),$$

where ζ is a real number and $\zeta\geq 1$.

Theorem 5. *Let $\sigma>-\alpha$. If $f(z)\in S_{\lambda,l}^{*m}(a,c,A;\alpha)$, then*

$$L_\sigma f(z)\in S_{\lambda,l}^{*m}(a,c,A;\alpha).$$

Proof. Suppose that $f(z)\in S_{\lambda,l}^{*m}(a,c,A;\alpha)$ and let

$$\frac{z\left(I_{\lambda,l}^m(a,c,A)L_\sigma f(z)\right)'}{I_{\lambda,l}^m(a,c,A)L_\sigma f(z)}=\frac{1+(1-2\alpha)\omega(z)}{1-\omega(z)}, \tag{57}$$

where $\omega(0)=0$. Then, by using (56) and (57), we have

$$\frac{I_{\lambda,l}^m(a,c,A)f(z)}{I_{\lambda,l}^m(a,c,A)L_\sigma f(z)}=\frac{\sigma+1+(1-\sigma-2\alpha)\omega(z)}{(\sigma+1)(1-\omega(z))}, \tag{58}$$

which, upon logarithmic differentiation, yields

$$\frac{z\left(I_{\lambda,l}^m(a,c,A)f(z)\right)'}{I_{\lambda,l}^m(a,c,A)f(z)}=\frac{1+(1-2\alpha)\omega(z)}{1-\omega(z)}+\frac{z\omega'(z)}{1-\omega(z)}+\frac{(1-\sigma-2\alpha)z\omega'(z)}{(1-\sigma-2\alpha)\omega(z)+\sigma+1}, \tag{59}$$

so that

$$\frac{z\left(I_{\lambda,l}^{m}(a,c,A)f(z)\right)'}{I_{\lambda,l}^{m}(a,c,A)f(z)}-\alpha=(1-\alpha)\frac{1+\omega(z)}{1-\omega(z)}+\frac{z\,\omega'(z)}{1-\omega(z)}+\frac{(1-\sigma-2\alpha)z\,\omega'(z)}{(1-\sigma-2\alpha)\,\omega(z)+\sigma+1}. \tag{60}$$

Now, assuming that $\max\limits_{|z|\le|z_0|}|\omega(z)|=|\omega(z_0)|=1\ (z\in U)$ and applying Jack's lemma, we obtain

$$z_0\omega'(z_0)=\zeta\omega(z_0)\quad (\zeta\in\mathbb{R},\zeta\ge 1). \tag{61}$$

If we set $\omega(z_0)=e^{i\theta}\ (\theta\in\mathbb{R})$ in (60) and observe that

$$Re\left\{(1-\alpha)\frac{1+\omega(z_0)}{1-\omega(z_0)}\right\}=0,$$

then, we obtain

$$\begin{aligned} Re\left\{\frac{z\left(I_{\lambda,l}^{m}(a,c,A)f(z)\right)'}{I_{\lambda,l}^{m}(a,c,A)f(z)}-\alpha\right\} &= Re\left\{\frac{z_0\omega'(z_0)}{1-\omega(z_0)}+\frac{(1-\sigma-2\alpha)z_0\omega'(z_0)}{(1-\sigma-2\alpha)\,\omega(z_0)+\sigma+1}\right\}\\ &= Re\left\{\frac{\zeta e^{i\theta}}{1-e^{i\theta}}+\frac{(1-\sigma-2\alpha)\zeta e^{i\theta}}{(1-\sigma-2\alpha)e^{i\theta}+\sigma+1}\right\}\\ &= -\frac{2\zeta(\sigma+\alpha)(1-\alpha)}{2(1+\sigma)(1-\sigma-2\alpha)\cos\theta+(1+\sigma)^2+(1-\sigma-2\alpha)^2}\\ &< 0, \end{aligned}$$

which obviously contradicts the hypothesis $f(z)\in S_{\lambda,l}^{*m}(a,c,A;\alpha)$. Consequently, we can deduce that $|\omega(z)|<1\ (z\in U)$, which, in view of (57), proves the integral-preserving property asserted by Theorem 5.

Taking $a=c$ in Theorem 5, we obtain the following corollary.

Corollary 3. *Let* $\sigma>-\alpha$. *If* $f(z)\in S_{\lambda,l}^{*m}(\alpha)$, *then* $L_\sigma f(z)\in S_{\lambda,l}^{*m}(\alpha)$, where the subclass $S_{\lambda,l}^{*m}(\alpha)$ is defined by (22).

Remark 6.

(i) Taking $\lambda=1$, $m=\mu\ (\mu>0)$ and $l=a-1\ (a>0)$ in Corollary 3, we obtain the results obtained by Aouf [1, Theorem 5];

(ii) Taking $\lambda=l=1$ and $m=\sigma\ (\sigma>0)$ in Corollary 3, we obtain the results obtained by Liu [19, Theorem 5].

Taking $m=0$ in Theorem 5, we obtain the following corollary.

Corollary 4. *Let* $\sigma>-\alpha$. *If* $f(z)\in S^{*}(a,c,A;\alpha)$, *then* $L_\sigma f(z)\in S^{*}(a,c,A;\alpha)$, where the subclass $S^{*}(a,c,A;\alpha)$ is defined by (26).

Remark 7. Taking $\alpha=0$, $a=\mu-1\ (\mu>0)$, $c=\lambda\ (\lambda>-1)$ and $A=1$ in Corollary 4, we obtain the results obtained by Choi et al. [9, Theorem 4, with $\varphi(z)=\frac{1+z}{1-z}$].

Theorem 6. *Let* $\sigma > -\alpha$. *If* $f(z) \in C_{\lambda,l}^{m}(a,c,A;\alpha)$, *then*

$$L_{\sigma}f(z) \in C_{\lambda,l}^{m}(a,c,A;\alpha).$$

Proof. By applying Theorem 5, it follows that

$$\begin{aligned} f(z) \in C_{\lambda,l}^{m}(a,c,A;\alpha) &\Leftrightarrow zf'(z) \in S_{\lambda,l}^{*m}(a,c,A;\alpha) \\ &\Rightarrow L_{\sigma}\left(zf'(z)\right) \in S_{\lambda,l}^{*m}(a,c,A;\alpha) \\ &\Leftrightarrow z\left(L_{\sigma}f(z)\right)' \in S_{\lambda,l}^{*m}(a,c,A;\alpha) \\ &\Leftrightarrow L_{\sigma}f(z) \in C_{\lambda,l}^{m}(a,c,A;\alpha), \end{aligned}$$

which proves Theorem 6.

Taking $a = c$ in Theorem 6, we obtain the following corollary.

Corollary 5. *Let* $\sigma > -\alpha$. *If* $f(z) \in C_{\lambda,l}^{m}(\alpha)$, *then* $L_{\sigma}f(z) \in C_{\lambda,l}^{m}(\alpha)$, where the subclass $C_{\lambda,l}^{m}(\alpha)$ is defined by (23).

Remark 8.

(i) Taking $\lambda = 1$, $m = \mu(\mu > 0)$ and $l = a-1(a > 0)$ in Corollary 5, we obtain the result obtained by Aouf [1, Theorem 6];

(ii) Taking $\lambda = l = 1$ and $m = \sigma(\sigma > 0)$ in Corollary 5, we obtain the result obtained by Liu [19, Theorem 6].

Taking $m = 0$ in Theorem 6, we obtain the following corollary.

Corollary 6. *Let* $\sigma > -\alpha$. *If* $f(z) \in C(a,c,A;\alpha)$, *then* $L_{\sigma}f(z) \in C(a,c,A;\alpha)$, where the subclass $C(a,c,A;\alpha)$ is defined by (27).

Remark 9. Taking $\alpha = 0$, $a = \mu - 1(\mu > 0)$, $c = \lambda(\lambda > -1)$ and $A = 1$ in Corollary 6, we obtain the results obtained by Choi et al.[9, Theorem 5, with $\varphi(z) = \frac{1+z}{1-z}$].

Theorem 7. *Let* $\sigma > -\alpha$. *If* $f(z) \in K_{\lambda,l}^{m}(a,c,A;\beta,\alpha)$, *then*

$$L_{\sigma}f(z) \in K_{\lambda,l}^{m}(a,c,A;\beta,\alpha).$$

Proof. Suppose that $f(z) \in K_{\lambda,l}^{m}(a,c,A;\beta,\alpha)$. Then, there exists a function $g(z) \in S_{\lambda,l}^{*m}(a,c,A;\alpha)$ such that

$$Re\left\{\frac{z\left(I_{\lambda,l}^{m}(a,c,A)f(z)\right)'}{I_{\lambda,l}^{m}(a,c,A)g(z)}\right\} > \beta.$$

Thus, upon setting

$$\frac{z\left(I_{\lambda,l}^{m}(a,c,A)L_{\sigma}f(z)\right)'}{I_{\lambda,l}^{m}(a,c,A)L_{\sigma}g(z)} - \beta = (1-\beta)h(z), \tag{62}$$

where $h(z)$ is given by (31), we find from (56) that

$$\frac{z\left(I_{\lambda,l}^{m}(a,c,A)f(z)\right)'}{I_{\lambda,l}^{m}(a,c,A)g(z)}=\frac{I_{\lambda,l}^{m}(a,c,A)zf'(z)}{I_{\lambda,l}^{m}(a,c,A)g(z)}$$

$$=\frac{z\left(I_{\lambda,l}^{m}(a,c,A)L_{\sigma}zf'(z)\right)'+\sigma\left(I_{\lambda,l}^{m}(a,c,A)L_{\sigma}zf'(z)\right)}{z\left(I_{\lambda,l}^{m}(a,c,A)L_{\sigma}g(z)\right)'+\sigma\left(I_{\lambda,l}^{m}(a,c,A)L_{\sigma}g(z)\right)}$$

$$=\frac{\dfrac{z\left(I_{\lambda,l}^{m}(a,c,A)L_{\sigma}zf'(z)\right)'}{I_{\lambda,l}^{m}(a,c,A)L_{\sigma}g(z)}+\sigma\dfrac{I_{\lambda,l}^{m}(a,c,A)L_{\sigma}zf'(z)}{I_{\lambda,l}^{m}(a,c,A)L_{\sigma}g(z)}}{\dfrac{z\left(I_{\lambda,l}^{m}(a,c,A)L_{\sigma}g(z)\right)'}{I_{\lambda,l}^{m}(a,c,A)L_{\sigma}g(z)}+\sigma}. \tag{63}$$

Since $g(z)\in S_{\lambda,l}^{*m}(a,c,A;\alpha)$, we know from Theorem 5 that $L_{\sigma}g(z)\in S_{\lambda,l}^{*m}(a,c,A;\alpha)$. Then, we can set

$$\frac{z\left(I_{\lambda,l}^{m}(a,c,A)L_{\sigma}g(z)\right)'}{I_{\lambda,l}^{m}(a,c,A)L_{\sigma}g(z)}-\alpha=(1-\alpha)H(z), \tag{64}$$

where $Re\{H(z)\}>0$. Then, substituting (62) and (64) into (63), we have

$$\frac{z\left(I_{\lambda,l}^{m}(a,c,A)f(z)\right)'}{I_{\lambda,l}^{m}(a,c,A)g(z)}=\frac{\dfrac{z\left(I_{\lambda,l}^{m}(a,c,A)L_{\sigma}zf'(z)\right)'}{I_{\lambda,l}^{m}(a,c,A)L_{\sigma}g(z)}+\sigma\left[\beta+(1-\beta)h(z)\right]}{\left[(1-\alpha)H(z)+\alpha\right]+\sigma}. \tag{65}$$

Also, we find from (62) that

$$z\left(I_{\lambda,l}^{m}(a,c,A)L_{\sigma}f(z)\right)'=\left(I_{\lambda,l}^{m}(a,c,A)L_{\sigma}g(z)\right)\left[(1-\beta)h(z)+\beta\right]. \tag{66}$$

Differentiating both sides of (66) with respect to z, we obtain

$$z\left(z\left(I_{\lambda,l}^{m}(a,c,A)L_{\sigma}f(z)\right)'\right)'=(1-\beta)zh'(z)\left(I_{\lambda,l}^{m}(a,c,A)L_{\sigma}g(z)\right)$$

$$+z\left(I_{\lambda,l}^{m}(a,c,A)L_{\sigma}g(z)\right)'\left[(1-\beta)h(z)+\beta\right], \tag{67}$$

that is,

$$\frac{z\left(z\left(I_{\lambda,l}^{m}(a,c,A)L_{\sigma}f(z)\right)'\right)'}{I_{\lambda,l}^{m}(a,c,A)L_{\sigma}g(z)}=(1-\beta)zh'(z)+\left[(1-\beta)h(z)+\beta\right]\left[(1-\alpha)H(z)+\alpha\right]. \tag{68}$$

Substituting (68) into (65), we find that

$$\frac{z\left(I_{\lambda,l}^{m}(a,c,A)f(z)\right)'}{I_{\lambda,l}^{m}(a,c,A)g(z)}-\beta=(1-\beta)h(z)+\frac{(1-\beta)zh'(z)}{\left[(1-\alpha)H(z)+\alpha\right]+\sigma}. \tag{69}$$

Then, by setting $u=h(z)=u_1+iu_2$ and $v=zh'(z)=v_1+iv_2$, we can define the function $\psi(u,v)$ by

$$\psi(u,v)=(1-\beta)u+\frac{(1-\beta)v}{\left[(1-\alpha)H(z)+\alpha\right]+\sigma},$$

where $(u,v)\in D=\mathbb{C}\times\mathbb{C}$. The remainder of our proof of Theorem 7 is similar to that of Theorem 3, so we choose to omit the analogous details involved.

Remark 10.

(i) Taking $m = s\left(s \in \mathbb{R}\right)$, $\lambda = 1$, $a = \mu - 1\left(\mu > 0\right)$, $c = 0$ and $A = 1$ in Theorems 5-7, we obtain the results obtained by Cho and Kim [8, Theorems 3.1-3.3 with $\varphi(z) = \psi(z) = \frac{1+z}{1-z}$];

(ii) Taking $a = \mu - 1\left(\mu > 0\right)$, $c = 0$ and $A = 1$ in Theorems 5-7, we obtain the results obtained by Aouf and El-Ashwah [3, Theorems 4-6 with $\varphi(z) = \psi(z) = \frac{1+z}{1-z}$];

(iii) Taking $l = 0$, $a = \mu - 1\left(\mu > 0\right)$, $c = 0$ and $A = 1$ in Theorems 5-7, we obtain the results obtained by Aouf et al. [2, Theorems 4-6 with $\varphi(z) = \psi(z) = \frac{1+z}{1-z}$].

Taking $a = c$ in Theorem 7, we obtain the following corollary.

Corollary 7. *Let* $\sigma > -\alpha$. *If* $f(z) \in K_{\lambda,l}^{m}(\beta,\alpha)$, *then* $L_\sigma f(z) \in K_{\lambda,l}^{m}(\beta,\alpha)$, where the subclass $K_{\lambda,l}^{m}(\beta,\alpha)$ is defined by (24).

Remark 11.

(i) Taking $\lambda = 1$, $m = \mu\left(\mu > 0\right)$ and $l = a - 1\left(a > 0\right)$ in Corollary 7, we obtain the results obtained by Aouf [1, Theorem 7];

(ii) Taking $\lambda = l = 1$ and $m = \sigma\left(\sigma > 0\right)$ in Corollary 7, we obtain the results obtained by Liu [19, Theorem 7].

Taking $m = 0$ in Theorem 7, we obtain the following corollary.

Corollary 8. *Let* $\sigma > -\alpha$. *If* $f(z) \in K(a,c,A;\beta,\alpha)$, *then* $L_\sigma f(z) \in K(a,c,A;\beta,\alpha)$, where the subclass $K(a,c,A;\beta,\alpha)$ is defined by (28).

Remark 12. Taking $\alpha = \beta = 0$, $a = \mu - 1\left(\mu > 0\right)$, $c = \lambda\left(\lambda > -1\right)$ and $A = 1$ in Corollary 8, we obtain the results obtained by Choi et al. [9, Theorem 6, with $\varphi(z) = \psi(z) = \frac{1+z}{1-z}$].

Theorem 8. *Let* $\sigma > -\alpha$. *If* $f(z) \in K_{\lambda,l}^{*m}(a,c,A;\beta,\alpha)$, *then*

$$L_\sigma f(z) \in K_{\lambda,l}^{*m}(a,c,A;\beta,\alpha).$$

Proof. Just as we derived Theorem 6 from Theorem 5. Easily, we can deduce Theorem 8 from Theorem 7. So we choose to omit the proof.

Taking $a = c$ in Theorem 8, we obtain the following corollary.

Corollary 9. *Let* $\sigma > -\alpha$. *If* $f(z) \in K_{\lambda,l}^{*m}(\beta,\alpha)$, *then* $L_\sigma f(z) \in K_{\lambda,l}^{*m}(\beta,\alpha)$, where the subclass $K_{\lambda,l}^{*m}(\beta,\alpha)$ is defined by (25).

Remark 13.

(i) Taking $\lambda = 1$, $m = \mu\left(\mu > 0\right)$ and $l = a - 1\left(a > 0\right)$ in Corollary 9, we obtain the results obtained by Aouf [1, Theorem 8];

(ii) Taking $\lambda = l = 1$ and $m = \sigma(\sigma > 0)$ in Corollary 9, we obtain the results obtained by Liu [19, Theorem 8].

Taking $m = 0$ in Theorem 8, we obtain the following corollary.

Corollary 10. *Let* $\sigma > -\alpha$. *If* $f(z) \in K^*(a,c,A;\beta,\alpha)$, *then* $L_\sigma f(z) \in K^*(a,c,A;\beta,\alpha)$, where the subclass $K^*(a,c,A;\beta,\alpha)$ is defined by (29).

4. CONCLUSION

From the results obtained in this paper, we can easily obtain the corresponding results of the well-known operators such as *Jung-Kim-Srivastava* operator, *Carlson-Shaffer* operator, *Noor* operator, *Ruscheweyeh* operator, *Salagean* operator, and others as a special choices of the parameters as mentioned before.

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