Proceedings of the Pakistan Academy of Sciences: A Physical and Computational Sciences 63(3): ------- (2026)
Copyright © Pakistan Academy of Sciences
ISSN (Print): 2518-4245; ISSN (Online): 2518-4253
http://doi.org/10.53560/PPASA(63-3)716

Pakistan Academy of Sciences

Research Article

On Finite Structures of Cyclicpoid Classes and Composed Functions in Star-Like Transformation Semigroups

Sulaiman Awwal Akinwunmi[1], Garba Risqot Ibrahim[2], Nabi Bux Kalhoro[3], Aijaz Ahmed Bhatti[4*], and Abdussamad Tanko Imam[5]

[1]Department of Mathematics and Statistics, Federal University of Kashere, Gombe, Nigeria
[2]Department of Mathematics and Statistics, Kwara State University Malete, Kwara, Nigeria
[3]Department of Mathematics, The University of Larkano, Larkana, Sindh, Pakistan
[4]Institute of Mathematics and Computer Science, University of Sindh, Jamshoro, Pakistan
[5]Department of Mathematics, Ahmadu Bello University Zaria, Kaduna, Nigeria

ACCEPTED MANUSCRIPT

Abstract: This research explores the algebraic and structural characteristics of finite star-like transformation semigroups that have cyclic features. We formalize the concept of cyclicpoid classes, characterized as star-like partial transformations that function alongside single-valued unary inversion operations. Concerning the structural deviation from classical cyclic groups and monogenic semigroups, specifically, we examine the systems where their kernels do not constitute group subsemigroups. By employing generator-index-period parameters (m, p) joint capacity parameters v^*, and subsemigroup kernel decompositions, we establish a lower-bound theorems that regulate operational transformation paths (Theorem 3). Additionally, by representing composition matrices in real min-plus algebras, we create a tropical polynomial form to ascertain exact tropical roots (eigenvalues). A non-group Cayley table with 4 elements, an evaluation of a 3×3 tropical matrix eigenvalues, and applications of combinatorial parameters are provided to validate the theoretical framework. This study integrates star-like partial function limits with tropical spectral descriptions for generalized cyclic transformation semigroups.

Keywords: Monogenic Semigroup, Cayley Table, Star-like Cyclicpoid, Transformation Semigroup, Tropical Algebra, Semigroup Kernel.

1. INTRODUCTION

Graph theory's close relationship to the algebraic theory of semigroups is one of its noteworthy applications. The relationship between star-like semigroups and the geometric-combinatorial theories underlying this algebraic representation has been studied, in particular, using the cyclicpoid class as a model. In the literature, there has been a notable increase in connections between tropical geometry, algebraic geometry, and related combinatorial models throughout the past fifteen years [1, 2]. For more recent work in tropical geometry, the reader is referred to Helminck and Ren [3], Puentes *et al.* [4], and Ibrahim *et al.* [5]. Tropical geometry requires exact algebraic definitions before meaningful mathematical progress can be made. This characteristic, which is often emphasized by the use of quotation marks, allowed Itenberg *et al.* [2] to differentiate tropical algebraic operations from their classical algebraic equivalents. While Katz [6] emphasizes the study of linear systems on graphs and lifting tropical curves in higher-dimensional spaces, Brugallé and Shaw [7] adhere to this convention. Similarly, Shaw [8] studies matroidal fans' tropical intersection products. These contributions demonstrate how tropical geometry is becoming increasingly varied. Recent studies by Usamot *et al.* [9], Idayat *et al.* [10], and Ibrahim *et al.* [11] have investigated the

Received: January 2026; Revised: August 2026; Accepted: September 2026
* Corresponding Author: Aijaz Ahmed Bhatti <aijaz.bhatti@usindh.edu.pk>

tropical characteristics of idempotent elements in the full transformation semigroup, offering new information on structural properties from a tropical viewpoint.

Classical algebraic semigroup theory provides the foundational apparatus for studying algebraic structures equipped with associative binary operations [25]. Central to this theory is the investigation of transformation semigroups and symmetric inverse semigroups, which model partial and full mappings on finite sets [15, 26]. A cornerstone of mainstream semigroup theory is the study of monogenic (or cyclic) semigroups $P\omega_n^* = \langle b \rangle$, whose fundamental structure governed by the generator index *m* and period *p* and associated Green's relations are thoroughly characterized in classic treatises [14, 25]. Order-preserving procedures and isometric mappings have further expanded the scope of semigroup theory. For instance, the ranks of specific semigroups of order-preserving partial isometries on a finite chain were examined by Ali *et al.* [12], while Garba [13] investigated idempotent ranks of semigroups of order-preserving transformations. Comprehensive analyses by Howie [14] and Ganyushkin and Mazorchuk [15] provide essential foundational references for finite transformation semigroups, complemented by Higgins [16] regarding order-preserving mappings.

Recently, Akinwunmi [17] introduced a novel class of star-like semigroups and characterized cyclicpoid functions, focusing on the detailed interconnections between group theory, semigroup theory, and tropical algebra. Akinwunmi *et al.* [18] standardized the co-existence relations of operator algebras and transformation semigroups, establishing a key link between group theory, semigroup theory, and functional analysis. Furthermore, Akinwunmi *et al.* [19] explored the geometric properties of the star-like transformation semigroup $C_y P\omega_n^*$, establishing a tropical graph by examining its algebraic and tropical structure. Structural properties of related semigroups and transformations have also been extensively studied by Imam and Ibrahim [20], Umar [21], and Kachalla and Madu [22], addressing parameters such as breadth, collapse, fix, height, waist, and nilpotent product representations. Recent developments in transformation semigroups have also focused on structural restrictions and specialized element properties. For instance, Srimora *et al*. [23] investigated the regularity of transformation semigroups with invariant sets restricted by equivalence relations, while Chinram *et al.* [24] systematically characterized left and right magnifying elements in generalized partial transformation semigroups.

1.1. Motivation and Research Gap

Star-like monogenic semigroups frequently display cyclic substructures; however, their overall configurations rarely create cyclic groups due to non-invertible transient trajectories and an absence of global identities. The cyclic core present in a larger star-like semigroup is denoted by the cyclicpoid class G_{cy}^*. Despite recent advancements in transformation semigroup theory, a notable structural gap persists: existing frameworks fail to create a direct connection between star-like partial transformations with unary inversion operations and min-plus tropical spectral representations [23, 24]. Furthermore, traditional static models fall short in handling dynamic alterations in index, period, and kernel across various composition paths, particularly when zero-adjunctions $(\emptyset)$ and non-group subsemigroup kernels are present.

1.2. Objectives and Main Contributions

To bridge this gap, this paper introduces a unified algebraic framework combining star-like cyclicpoids $(C_y P\omega_n^*)$ with max-plus/min-plus spectral analysis. The primary contributions of this paper are structured as follows:

(i) Formalization of Cyclicpoid Axiomatics: We establish exact operational definitions for star-like cyclicpoid classes operating under unary inversion laws (-1) and zero-adjunction rules, establishing key inverse operational duality theorems (Section 2, Definition 2).

(ii) Tropical Spectral Representation: We map star-like composition matrices onto real min-plus algebras, providing a tropical polynomial characterization to determine precise tropical roots (α_i^*) and invariant eigenvalues for transformation matrices (Section 3.1).

(iii) Combinatorial Lower Bounds: We prove explicit lower bounds for operational transformation paths and spinnable cyclicpoid bijections (Theorem 3) derived from generator-index-period parameters (m, p) and joint capacity

ACCEPTED MANUSCRIPT

parameters v^* (Section 3.3).

(iv) Concrete Mathematical Illustrative Examples: We present a non-group 4-element star-like monogenic semigroup Cayley table (Example 1), a 3×3 tropical matrix eigenvalue evaluation (Example 2), and folded combinatorial parameter evaluations (Examples 5–8) to validate the theoretical model.

2. PROPOSED WORK

To ensure mathematical clarity, rigor, and consistent presentation throughout the formal development of our proposed framework, the primary symbols, parameters, and algebraic operators used in this paper are defined and summarized in Table 1.

The theory of star-like transformation semigroups is expanded in this work, with a focus on tropical and combinatorial frameworks, building on these foundations. The results are intended to pave the way for the application of semigroup-theoretic methods to computational algebra and other branches of mathematics. One can place the cyclicpoid class concept of the star-like semigroup $\boldsymbol{G_{cy}^*}$ in a hierarchy of cyclic, tricyclic, polycyclic, and related subsemigroups by admitting a characterization in terms of its index, period, and kernel. With this structural framework, we study combinatorial growth properties, recursive identities, and product decompositions were studied. This work develops exact algebraic relations for classes of spinnable cyclicpoids. Thus, $\boldsymbol{G_{cy}^*}$ is described in this study in terms of index, period, and kernel, along with connections to traditional cyclic structures, creates novel combinatorial identities and inequalities for spinnable cyclicpoid classes, expanded the findings using explicit recursive and algebraic results, and presented fundamental definitions and axioms for star-like cyclicpoid classes.

However, Ganyushkin and Mazorchuk's [15] and Higgins' [16] classical approaches in transformation semigroups form the basis for finite transformations, they clearly demonstrate structural constraints when dealing with complex algebraic structures. In particular, conventional models are limited by static constraints that do not take into account dynamic index, period, and kernel shifts among different composition routes. Furthermore, conventional subsemigroup structures continue to face difficulties in reliably managing unary inversion operations within non-group structures and recognizing recursive spinnable transformations. Although, path compositions linked to early-stage invertibility and multiplicity tracking by Akinwunmi *et al.* [19] and modern transformation investigations by Imam and Ibrahim [20] employ advanced foundational parameters. Our suggested method uses special unary operations of the cyclicpoid class G_{cy}^* and the star-like partial cyclic semigroup $C_yP\omega_n^*$ to solve these specific structural flaws. By explicitly classifying finite algebraic structures according to their intrinsic structural restrictions, this method effectively overcomes the strict constraints of traditional transformation mappings.

Definition 1 (Monogenic Star-Like Semigroup) [14, 15]: Let $P\omega_n^* = \langle b \rangle$ be a finite monogenic

Table 1. Standardized Mathematical Notation Used Throughout the Manuscript.

Notations	Formal definition	Scope
$\boldsymbol{C_yP\omega_n^*}$	Star-like cyclicpoid partial transformation semigroup on finite chains	Algebraic structure
$\boldsymbol{m, p}$	Generator index $(m \geq 1)$ and period $(p \geq 1)$ of monogenic kernel	Kernel parameters
$\boldsymbol{K_{\beta^*}}$	Unique star-like kernel subsemigroup $K_{\beta^*} = \{\beta^{m*}, \beta^{(m+1)*}, \beta^{(m+2)*}, \dots \beta^{(m+p-1)*}\}$	Subsemigroup
$\boldsymbol{\alpha_i^*}$	Invariant tropical spectral root (matrix eigenvalue)	Min-plus spectrum
$\boldsymbol{f(n, C_yP\omega_n^*)}$	Combinatorial lower bound function for transformation counts	Combinatorics
$\boldsymbol{v^*}$	Joint capacity parameter governing operational paths	Combinatorics
$\emptyset$	Zero-adjunction element	Combinatorics

ACCEPTED MANUSCRIPT

star-like semigroup generated by a single element $\beta^* = b$. The index set D_{β^*} governing the generator iterates β_k^* is defined as the finite subset of natural numbers:

$$D_{\beta^*} = \{1,2,\dots,m+p-1\} \subset N^+,$$

where, $m \geq 1$ denotes the index and $p \geq 1$ denote the period. The algebraic structure $C_y P\omega_n^* = \{\beta^*{}_k : k \in D_{\beta^*}\}$ under the operational concatenation $(\cdot)$ is isomorphic to the additive integer semigroup $(N^+{}_{m,p}, +_{m,p})$. Here, the underlying set is

$$N^+{}_{m,p} = \{1, 2, \dots, m+p-1\},$$

and the binary operation $\mathrm{N}^+{}_{m,p}$ is defined by

$$i +_{m,p} j = \begin{cases} i+j, & if\ \ i+j < m+p \\ m + ((i+j-m) mod p, & if\ \ i+j \geq m+p \end{cases},$$

under the isomorphism mapping $\varphi: \beta^*{}_k \to k$.

Example 1: Let $P\omega_n^* = \{b, b^2, b^3, b^4\}$ be a finite monogenic semigroup generated by β^* subject to $\beta^5 = \beta^3$. Here, the index is $m = 3$, the period is $p = 2$, and the kernel $K_{\beta^*} = b^3, b^4$ forms a subsemigroup that is not a group (lacking an identity element for all elements of $P\omega_n^*$), distinguishing it from a standard cyclic group. The operational behavior of $P\omega_n^*$ is explicitly illustrated in Table 2.

Definition 2 (Cyclicpoid Class) [19]: Let $G_{cy}^* \in C_y P\omega_n^*$ be a non-empty set equipped with a single- valued unary operation $(^{-}1): G_{cy}^* \to G_{cy}^*$ and a partial binary operation $(\cdot): D \to G_{cy}^*$, where $D \subseteq G_{cy}^* \times G_{cy}^*$ represents the domain of defined star-like composable pairs (α^*, β^*). The structure $(G_{cy}^*, \cdot, -1)$ forms an inverse cyclicpoid if it satisfies the following explicit axioms for all star-like composable transformations $\alpha^*, \beta^*, \gamma^* \in G_{cy}^*$:

Table 2. Cayley Table of the Finite Star-like Monogenic Semigroup.

$\boldsymbol{\beta^*(P\omega_n^*)}$	$\boldsymbol{b}$	$\boldsymbol{b^2}$	$\boldsymbol{b^3}$
$\boldsymbol{b}$	b^2	b^3	b^4
$\boldsymbol{b^2}$	b^3	b^4	b^3
$\boldsymbol{b^3}$	b^4	b^3	b^4
$\boldsymbol{b^4}$	b^3	b^4	b^3

(i) Partial associativity:
$(\alpha^* \cdot \beta^*) \cdot \gamma^* = \alpha^* \cdot (\beta^* \cdot \gamma^*)$ whenever either side is defined.
(ii) Regular inversion identities: For every $\alpha^* \in G_{cy}^*$ the single-valued image. $\alpha^{-1*} \in G_{cy}^*$ satisfies $\alpha^* \cdot \alpha^{-1*} = \alpha^*$ and $\alpha^{-1*} \cdot \alpha^* \cdot \alpha^{-1*} = \alpha^{-1*}$.
(iii) Commutative idempotents: For all star-like idempotents $e^*, f^* \in E(G_{cy}^*)$, where:
$E(G_{cy}^*) = \{x^*, x^{-1*} : x \in G_{cy}^*\}$, $\quad e^* \cdot f^* = f^* \cdot e^*$.
(iv) Operational inversion duality: For any star-like composable transformation pair $(\alpha^*, \beta^*) \in D$, the anti homomorphic involution law holds strictly: $(\alpha^* \cdot \beta^*)^{-1} = \beta^{*-1} \cdot \alpha^{*-1}$.

Definition 3 (Semigroup with Adjoined Zero): Let $(P\omega_n^*, \cdot)$ be a finite monogenic star-like semigroup. If $P\omega_n^*$ contains a transformation $\emptyset \in P\omega_n^*$ such that $0 \cdot x = x$, and $\emptyset = 0$ for all $x \in P\omega_n^*$, then $\emptyset$ is the unique partial element of the semigroup. If no such absorbing element exists, a unique partial element denoted by $\emptyset$ is formally adjoined to create $P\omega_n^{*0} = P\omega_n^* \cup \{\emptyset\}$, subject to explicit operation rules: $\emptyset \cdot \alpha^* = \alpha^* \cdot \emptyset = \emptyset$ for all $\alpha^* \in P\omega_n^{*0}$. This explicit adjunction guarantees that $P\omega_n^{*0}$ forms a semigroup with a uniquely defined absorbing partial (zero) transformation.

Definition 4 (Transformation Image and Annihilator Subsemigroups): Let $P\omega_n^{*0}$ be a finite star-like monogenic semigroup with adjoined zero element $\emptyset$. For a given transformation element $\alpha^* \in P\omega_n^{*0}$:
(i) The Range (Image Set) is defined as:

$$Im(\alpha^*) = \{\beta^* \cdot \alpha^* \cdot \beta^* \in P\omega_n^{*0}\} \subseteq P\omega_n^{*0}$$

(ii) The Right Annihilator Set is defined as:

$$Ann_r(\alpha^*) = \{u^* \in P\omega_n^{*0} : u^* \cdot \alpha^* = \emptyset\}.$$

Lemma 1: Let $P\omega_n^{*0}$ be a finite star-like monogenic semigroup, then the right annihilator set $Ann_r(\alpha^*)$ forms a valid subsemigroup N_{α^*} of $P\omega_n^{*0}$.
Proof. Let $u^*, v^* \in Ann_r(\alpha^*)$. By definition, $u^* \cdot \alpha^* = \emptyset$ and $v^* \cdot \alpha^* = \emptyset$. Utilizing the associativity of $P\omega_n^{*0}$ and the absorbing property of $\emptyset$ established in Definition 3 gives
$(u^* \cdot v^*) \cdot \alpha^* = u^* \cdot (v^* \cdot \alpha^*) = u^* \cdot \emptyset = \emptyset$.
Thus, $u^* \cdot v^* \in Ann_r(\alpha^*)$. Therefore, $Ann_r(\alpha^*)$ is closed under multiplication and constitutes a formal subsemigroup $N_{\alpha^*} \subseteq P\omega_n^{*0}$.

Definition 5: Let $G_{cy}^* = \{\alpha^*{}_k : k \in D_{\beta^*}\}$ be a finite cyclicpoid where each transformation

ACCEPTED MANUSCRIPT

$\alpha^*{}_k$ is indexed by $k_i \in D_{\beta^*} \subset N^+$. Let $W = \{w_1, w_2, \dots, w_n\} \subset D_{\beta^*}$ be a sequence of reference indices for $i \in \{1,2,\dots,n-1\}$. The structure G^*_{cy} satisfies the star-like index bounding condition if the integer index distances within the domain $(\mathrm{N}^+, |\cdot|)$ obey:
$|\alpha^*\, k_i - w_{i+1}| \le |\alpha^*\, k_{i+1} - w_i|$,
for all $i \in \{1,2,\dots,n-1\}$. Because k_i, w_i represent natural numbers in D_{β^*}, operational invariance under semigroup composition $\alpha^*{}_{k_i} \cdot\ \alpha^*{}_m = \alpha^*{}_{k_i +_{m,p} m}$ follows directly from the monotonicity of modular index addition $+_{m,p}$ defined on D_{β^*}.

Definition 6 (Matrix Transformation Semigroup): Let $M_n(\mathbb{R})$ be the set of $n \times n$ real matrices. The star-like matrix subsemigroup M* generated by a matrix $A^* \in M_n(\mathbb{R})$ is defined strictly under the binary operation of matrix composition/ multiplication $(\cdot)$: $M^* = \langle A^* \rangle = \{(A^*)^k : k \in D_{\beta^*}\}$. The algebraic structure $(M^*, \cdot)$ forms a multiplicative subsemigroup of $(M_n(\mathbb{R}), \cdot)$. While M^* resides within the vector space $M_n(\mathbb{R})$, M^* itself is not required or claimed to be closed under vector addition or scalar multiplication. All semigroup properties, identities, and annihilators defined on $M^*$are governed solely by matrix multiplication.

Definition 7: Let $\beta^* \in C_yP\omega^*_n$ be a spinnable star-like transformation semigroup on space V then;
(i) Collapse index (q^*) is defined as the codimension of the non-invertible transient kernel under β^* denoted by $q^*(\beta^*) = n - |Im(\beta)^*|$ representing the net state condensation as transformations paths collapse.
(ii) Relapse index $r^*(\beta^*)$ is defined as the minimal recurrence period required for a collapsed transformation to re-enter a structural cycle head, denoted by:

$$r^*(\beta^*) = min\{k \ge 1 | \beta^{*k}(x) \in Cycl(\beta^*) \text{ for all } x \in V\}.$$

These indices serve as structural parameters evaluated by the star-like operational lower and upper combinatorial functions in Theorems 5 and 6.

3. RESULTS AND DISCUSSION

The analytical and geometric outputs generated by the star-like partial cyclic semigroup $C_yP\omega^*_n$ are thoroughly examined in this section. We investigate the algebraic partitions, structural constraints, and operational behaviors of the cyclicpoid class G^*_{cy} under different composition lengths. These results are arranged into two main dimensions to guarantee structural clarity: the geometric partition bounds and recursive formulations represented by directional path configurations and the algebraic characterization of cyclicpoid classes via index, period, and kernel properties. We explicitly show how our model manages multi-directional structural flows over finite spaces by framing our results inside these real parameters.

3.1. The $f(n, C_yP\omega^*_n)$ Transformation's Star-like Generated Functions

Assume that $Z_n = \{1,2,3,\dots\}$ is the non-degenerated star-like number. Then, if M^* is an ordered, composed matrix of the star-like structure, we have:

$$|C_yP\omega^*_n| = \begin{cases} \alpha^*_{i,n} - D(\alpha^*_{i,n}) \\ v^*_{i,n} - n : v^* \in V^* \\ M^* : v^* \to v^* : n \times n, \end{cases} \quad (1)$$

such that:

$$M^* = \begin{bmatrix} (v^*_{i,1}, v^*_{j,1}, v^*_{k,1}) & \cdots & (v^*_{i,n}, v^*_{j,n}, v^*_{k,n}) \\ \vdots & \ddots & \vdots \\ (v^*_{i,n}, v^*_{j,n}, v^*_{k,n}) & \cdots & (v^*_{n,n}, v^*_{n,n}, v^*_{n,n}) \end{bmatrix},$$

with $i, j, k \in I(\alpha^*_i)$, $m \times m \in M^*$, and $v^* \in V^*$. Thus, a star-like composed matrix is a set $m \in M^*$ with the operations "+" and "·" by real numbers in which the operations must produce star-like composed matrices, such that the following are true:
(i) If α^*, β^* are composed matrices in M* then, $\alpha^* + \beta^*$ is a matrix in M^*.
(ii) If $\alpha^*$is a composed matrix in M^* and c is a scalar in $\mathbb{R}$ then, $c\alpha^*$is a composed matrix in M^*.
Assume

$$UT^*(x,n) = \sum_{i,j} a_{i,j} x^i n^j = Max^d_{i,j}(ix + jn + a_{i,j}), \quad (2)$$

that a tropical polynomial of degree d in two variables is a star-like function. In each term of the polynomial, i and j are integers that fulfill $i + j = d$ and range from 0 to d_0. By using the operations " + " and "." by real numbers on the star-like composition matrices $m \in M^*$, we get the tropical characterization of star-like transformation $C_yP\omega^*_n$ semigroups:

Example 2: Consider the 3×3 star-like transformation matrix M^*evaluated over the max-

ACCEPTED MANUSCRIPT

plus algebra $(\mathbb{R} \cup \{-\infty\}, max, +)$.

$$M^* = \begin{bmatrix} 0 & 2 & 3 \\ -\infty & 0 & 1 \\ 4 & -\infty & 0 \end{bmatrix}, \; M^* \in C_y P\omega_n^*$$

Compute

$$f(\alpha_i^*) = \max\{3\alpha_i^*, 2\alpha_i^* + 2, 2\alpha_i^* + 3, \alpha_i^* + 5, 7\},$$

Such that

$$3\alpha_i^* = 2\alpha_i^* + 2 \quad \Rightarrow \alpha_i^* = 2,$$
$$2\alpha_i^* + 3 = \alpha_i^* + 5 \quad \Rightarrow \alpha_i^* = 2,$$
$$\alpha_i^* + 5 = 7 \quad \Rightarrow \alpha_i^* = 2$$

Since the tropical characteristic polynomial of M^* is given by: $P_{max}(\gamma^*) = det_{trop}(\gamma^* \cdot I \oplus M^*)$. Then, evaluating the max-plus power expansions yields a piecewise linear convex function $P_{max}(\gamma^*)$. The non-smooth corner points (breakpoints) of $P_{max}(\gamma^*)$ where at least two affine-linear terms coincide define the max-plus eigenvalues. For the specific structural matrix M^* in this example, equating two coinciding star-like affine-linear transformations yields the dominant root $\alpha_i{}^* = 2$. In general max-plus systems, characteristic polynomials admit a spectrum of multiple break points and root multiplicities; uniqueness in this instance is an algebraic property of the specific cyclicpoid structure under study rather than a general tropical principle.

This establishes the role of tropical values in evaluating semigroup components. The algebraic constraints and structural features calculated here aim to address a particular drawback of conventional finite transformation semigroup models, namely the incapacity to translate dynamic changes across non-linear composition orbits, in order to contextualize these results within earlier work. Our model circumvents the strict boundary constraints included in conventional, static transformation mappings by analyzing the behaviors of the star-like partial cyclic semigroup. Thus, these results elaborate on how tropical algebraic restrictions directly determine the structural constraints and product lengths within constructed star-like transformations, building on the fundamental features addressed by Usamot *et al.* [9] and Ibrahim *et al.* [11]. This makes it possible to map algebraic partitions directly to the cyclicpoid classes' underlying structural stability. Additionally, our approach offers an alternate method for defining algebraic partitions, whereas the conventional rankings and transformation metrics developed by Howie [14] and Garba [13] mostly concentrate on linear chains. In a similar vein, this framework extends the nilpotent sequence limits studied by Kachalla and Madu [22] and the structural parameter approaches used by Umar [21], providing a more direct mapping of algebraic partitions to the underlying structural stability of the cyclicpoid classes without running into the boundary constraints present in standard partial transformations. In addition to the particular isometry bounds covered by Ali *et al.* [12], this allows a robust classification.

Lemma 2: Let $P\omega_n^{*0}$ be a finite star-like transformation semigroup adjoined with zero element (partial) $\emptyset$, and let $M^* \in M_n(\mathbb{R}_{max})$ be a matrix generator of $P\omega_n^{*0}$. Then, for any $\alpha_i{}^* \in \mathbb{R}$, star-like tropical root corresponding to $P_{max}(\gamma^*)$, its local tropical multiplicity given as:

$$\mu(\alpha_i{}^*) = k_1 - k_2 \in \{1,2,\dots,n\} \text{ where } k_1, k_2 \in \{0,1,\dots,n\},$$

with $k_1 < k_2$ are degree of γ^* in the dominant affine-linear transformations of $P_{max}(\gamma^*)$.

Proof. Suppose $M^* \in M_n(\mathbb{R}_{max})$ be a matrix generator of $P\omega_n^{*0}$. Define the star-like tropical characteristic polynomial $P_{max}(\gamma^*)$ over permutations $\sigma \in P\omega_n^{*0}$:

$$P_{max}(\gamma^*) = det_{trop}(\gamma^* \cdot I \oplus M^*) = \bigoplus_{P\omega_n^{*0}} \left(\sum_{j=1}^{n} (\gamma^* \cdot I \oplus M^*)_{j,\sigma(j)} \right).$$

From Equation (1), tropical star-like transformations was generated:

$$\alpha^* = \begin{array}{c} x^2 \\ x \\ 1 \end{array} \begin{pmatrix} \alpha_{11}^* & \alpha_{12}^* & \alpha_{13}^* \\ \alpha_{21}^* & \alpha_{22}^* & \alpha_{23}^* \\ \alpha_{31}^* & \alpha_{32}^* & \alpha_{33}^* \end{pmatrix}$$

$$\beta^* = \begin{array}{c} x^2 \\ x \\ 1 \end{array} \begin{pmatrix} \beta_{11}^* & \beta_{12}^* & \beta_{13}^* \\ \beta_{21}^* & \beta_{22}^* & \beta_{23}^* \\ \beta_{31}^* & \beta_{32}^* & \beta_{33}^* \end{pmatrix}, \text{ and}$$

$$\gamma^* = \begin{array}{c} x^2 \\ x \\ 1 \end{array} \begin{pmatrix} \gamma_{11}^* & \gamma_{12}^* & \gamma_{13}^* \\ \gamma_{21}^* & \gamma_{22}^* & \gamma_{23}^* \\ \gamma_{31}^* & \gamma_{32}^* & \gamma_{33}^* \end{pmatrix},$$

where $|P\omega_n^*, (\gamma^*)| > |P\omega_n^*, (\beta^*)| > |P\omega_n^*, (\alpha^*)|$. Then, using Equation (2)

$$T_{\alpha^*(x)} = \alpha_{11}^* x^4 + \alpha_{12}^* + \alpha_{21}^*)x^3 + (\alpha_{13}^* + \alpha_{22}^* + \alpha_{31}^*)x^2 + (\alpha_{23}^* + \alpha_{32}^*)x + \alpha_{33}^*$$

$$T_{\beta^*(x)} = \beta_{11}^* x^4 + \beta_{12}^* + \beta_{21}^*)x^3 + (\beta_{13}^* + \beta_{22}^* + \beta_{31}^*)x^2 + (\beta_{23}^* + \beta_{32}^*)x + \beta_{33}^*$$

$$T_{\gamma^*(x)} = \gamma_{11}^* x^4 + \gamma_{12}^* + \gamma_{21}^*)x^3 + (\gamma_{13}^* + \gamma_{22}^* + \gamma_{31}^*)x^2 + (\gamma_{23}^* + \gamma_{32}^*)x + \gamma_{33}^*$$

Was used to obtain a point wise maximum over a

ACCEPTED MANUSCRIPT

finite collection of affine-linear transformations:

$$P_{max}(\gamma^*) = \max_{0 \le k \le n} (k\,\gamma^* + c_k), \quad c_k \in \mathbb{R} \cup \{-\infty\},$$

where the integer coefficient $k \in \{0,1,\ldots,n\}$ denotes the degree of γ^* and c_k represents the maximum weight contribution from M^* for transformations of degree k. Since $\mathrm{P}_{\max}(\gamma^*)$ is the point wise maximum of finitely many affine-linear transformations, we observed that for $\gamma^* \in (\alpha_i{}^* - \epsilon, \alpha_i{}^*)$, $\mathrm{P}_{\max}(\gamma^*)$ is strictly dominated by a single affine transformation under $k_1\gamma^* + ck_1$ with degree $k_1 \in \{0,1,\ldots,n\}$ and for $\gamma^* \in (\beta_i{}^* - \epsilon, \beta_i{}^*)$, $\mathrm{P}_{\max}(\gamma^*)$ transitions to a distinct dominant affine transformation under $k_2\gamma^* + ck_2$ with degree $k_2 \in \{0,1,\ldots,n\}$. But at the boundary root, $\gamma^* = \alpha_i{}^*$, such that $k_1\alpha_i{}^* + ck_1 = k_2\beta_i{}^* + ck_2$. By the star-like convexity of the point wise maximum, the degree of the dominant transformation must strictly increase across the breakpoints giving $k_1 < k_2$ to obtain the tropical root multiplicity $\mu(\alpha_i{}^*) = k_2 - k_1$. Thus, if $k_1 < k_2$1with $k_1, k_2 \in \{0,1,\ldots,n\}$, $\mu(\alpha_i{}^*) = k_2 - k_1$ is well define integer in $\{1,2,\ldots,n\}$. Therefore, the active transformation constants ck_1, ck_2 depend directly on the on the entries of the semigroup generator $\mathrm{M}^* \in M_n(\mathbb{R}_{\max})$ where $\mu(\alpha_i{}^*)$ is local to $\alpha_i{}^*$ varies across different cyclicpoid matrix composition.

Lemma 3: Let $(\mathbb{R}_{max}, \oplus, \otimes)$ denote the max-plus semiring where $\alpha^* \oplus \beta^* = max(\alpha^*, \beta^*)$ and $\alpha^* \otimes \beta^* = \alpha^* + \beta^*$. Let $M^* \in M_n(\mathbb{R}_{max})$ be an idempotent star-like matrix satisfying $M^* \otimes M^* = M^*$. If $v \in \mathbb{R}^n \setminus \{-\infty\}^n$ is a nontrivial tropical eigen vector satisfying $M^* \otimes v = \gamma^* \otimes v$ for $\gamma^* \in \mathbb{R}$, then $\gamma^* = 0$.

Proof. Suppose $M^* \in M_n(\mathbb{R}_{max})$ is an idempotent star-like matrix. If M^* admits a tropical eigenvalue $\gamma^* \in \mathbb{R}$ with associated no-trivial eigenvector v. Consider a finitely many star-like matrices $M_i{}^* = M^*$:

$$M_i^* = \begin{array}{c} \\ x^{n-1} \\ x^{n-2} \\ x^{n-n} \\ 1 \end{array} \begin{array}{c} \begin{array}{cccc} x^{n-1} & x^{n-2} & x^{n-n} & 1 \end{array} \\ \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{1n} \\ k_{21} & k_{22} & k_{23} & k_{2n} \\ k_{31} & k_{32} & k_{33} & k_{3n} \\ k_{n1} & k_{n2} & k_{n3} & k_{nn} \end{pmatrix} \end{array}.$$

Then, applying $M_i{}^*$ to both sides of the eigenvalue equation and apply partial associativity (i) in Definition 2 yields

$$M_i{}^* \otimes (M_i{}^* \otimes v) = M_i{}^* \otimes (\gamma^* \otimes v) = \gamma^* \otimes (M_i{}^* \otimes v) = \gamma^* \otimes (\gamma^* \otimes v) = (2\gamma^*) \otimes v.$$

By star-like matrix idempotency $(M_i{}^* \otimes M_i{}^* = M_i{}^*)$, such that

$$(M_i{}^* \otimes M_i{}^*) \otimes v = M_i{}^* \otimes v = \gamma^* \otimes v.$$

Equating both sides gives $(2\gamma^*) \otimes v = \gamma^* \otimes v$. Since, v is non-trivial, there exists at least one component $k \in \{0,1,\ldots,n\}$ such that $v_k \neq -\infty$. Therefore, $2\gamma^* + v_k = \gamma^* + v_k$, implies $2\gamma^* = \gamma^* = 0$.

Lemma 4: Let $(A^*, B^* \in M_n(\mathbb{R}_{max}))$ matrix representations of star-like compositions over the max-plus semiring $(\mathbb{R} \cup \{-\infty\}, max, +)$ such that $tr_{trop}(\mathrm{M}^*) = \sum_{i=1}^{n} M_i{}^* = \max_{1 \le i \le n} M_i{}^*$. Then,

(i) $tr_{trop}(\mathrm{A}^*) = tr_{trop}(\mathrm{B}^*)$ if and only $\max_{1 \le i \le n} \mathrm{A}^*{}_i = \max_{1 \le i \le n} \mathrm{B}^*{}_i$.

(ii) $tr_{trop}(A^* \otimes B^*) = tr_{trop}(B^* \otimes A^*)$, for any two star-like composition matrices $A^*, B^* \in M_n(\mathbb{R}_{max})$.

Proof.

(i) Assume $A^*{}_{i,j} \le B^*{}_{i,j}$ for all $1 \le i, j \le n$ is limited to the diagonal structures of star-like composed matrices. Then, $\mathrm{A}^*{}_{i,i} \le \mathrm{B}^*{}_{i,i}$, for each $i \in \{1,\ldots,n\}$. Taking the maximum yields:

$$tr_{trop}(A^*) = \max_{1 \le i \le n} A^*{}_{i,i} \le \max_{1 \le i \le n} B^*{}_{i,i} = tr_{trop}(B^*).$$

(ii) By definition, $(A^* \otimes B^*)_{i,i} = \sum_{j=1}^{n} (A^*{}_{i,i} \otimes B^*{}_{j,i})$
$$= \max_{1 \le j \le n} (A^*{}_{i,i} \otimes B^*{}_{j,i})$$

Evaluating the traces yields:

$$tr_{trop}(A^* \otimes B^*) = \max_{1 \le i \le n} \left(\max_{1 \le j \le n} (A^*{}_{i,i} + B^*{}_{i,i}) \right)$$
$$= \max_{1 \le i,\ j \le n} (A^*{}_{i,i} + B^*{}_{i,i}).$$

Similarly,

$$tr_{trop}(B^* \otimes A^*) = \max_{1 \le j,\ i \le n} (B^*{}_{j,i} + A^*{}_{i,j}).$$

Example 3: Consider the entry wise ordered max-plus matrices $A^* \le B^* \le C^*$ such that

$$A^* = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}, \quad B^* = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}, \quad C^* = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$$

By Lemma 2, we obtain

$tr_{trop}(A^*) = max\,\{0,2\} = 2,$
$tr_{trop}(B^*) = max\,\{1,2\} = 2,$
$tr_{trop}(C^*) = max\,\{2,4\} = 4.$

Obviously, for $B^* \le C^*$ the star-like diagonal holds strictly since $tr_{trop}(B^*) = 2 < 4$. Therefore, the monotonicity increases as the diagonal entries increase.

ACCEPTED MANUSCRIPT

3.2 The Cyclicpoid Class G_{cy}^{*} of $f(n, C_yP\omega_n^{*})$

Let $\{U_i : i \in I\}$ (with $I \neq 0$) be a star-like partial semigroup of $C_yP\omega_n^{*}$, and let $\emptyset \neq I = \cap_{i=I} U_i$ be a star-like subsemigroup of $C_yP\omega_n^{*}$. When β^{*} is a non-empty star-like cyclicpoid transformation of $C_yP\omega_n^{*}$, then β^{*}, ($C_yP\omega_n^{*}$ itself), is contained in at least one subsemigroup of $C_yP\omega_n^{*}$. $\langle\beta^{*}\rangle$ is the singleton element $\{\beta^{*}\} \in C_yP\omega_n^{*}$ that generates the intersection of all star-like cyclicpoid subsemigroups of $C_yP\omega_n^{*}$ containing β^{*}. Two transformation options are available for $\langle\beta^{*}\rangle$. $\langle\beta^{*}\rangle$ has th$\Longrightarrow$llowing element:$\{\beta^{*}, \beta^{2*}, \beta^{3*}, \ . \ . \ .\}$.

(i) $\beta^{m*} = \beta^{p*} \rightarrow m = p \ , \beta^{*} \subseteq \beta_j^{*} (j \in I)$
(ii) If U^{*} is a subsemigroup of $C_yP\omega_n^{*}$ containing β^{*} then $\langle\beta^{*}\rangle = U_j^{*}$ for each $j \in I \ , \exists \ m \neq p : \ \beta^{m*} = \beta^{p*}$.

In the former instance, $\langle\beta^{*}\rangle$ is isomorphic to the natural number addition. In this situation, the element $\langle\beta^{*}\rangle$ is said to have infinite order, and $\langle\beta^{*}\rangle$ is an infinite star-like cyclicpoid transformation. Assume that in the latter case, there are two smallest positive integers m and p (the star-like index and star-like period of any given cyclicpoid transformation in $C_yP\omega_n^{*}$) such that $\beta^{m*} = \beta^{x*}$ for some positive integer $x \neq m$ and that $\beta^{m*} = \beta^{(m+p)*}$. We can also discuss the subsemigroup of $C_yP\omega_n^{*}$ produced β^{*} by, which will always contain at least one idempotent set if $C_yP\omega_n^{*}$ is a monoid. Since $m + p - 1$ is the definition of the order of β^{*}, the star-like period and the star-like index must meet the following axioms:

(i) $\beta^{m*} = \beta^{(m+p)*}$
(ii) $\beta^{(m+x)*} = \beta^{(m+y)*}$, if and only if $m + x \equiv m + y (mod \ p)$
(iii) $\langle\beta^{*}\rangle = \{\beta^{m*}, \beta^{2*}, \beta^{3*}, \ . \ . \ . \beta^{(m+p-1)*}\}$
(iv) $K_{\beta^{*}} = \{\beta^{m*}, \beta^{(m+1)*}, \beta^{(m+2)*}, \ . \ . \ . \beta^{(m+p-1)*}\}$, where $K_{\beta^{*}}$ is called the star like kernel of $\langle\beta^{*}\rangle$ and it is cyclicpoid.

The structure of the star-like $C_yP\omega_n^{*}$ cyclicpoid semigroups is determined by the pair (m, p). This is represented as $C_yP\omega_n^{*}(m, p)$, such that $C_yP\omega_n^{*}(1, p)$ is the cyclicpoid of order p. The novel class of transformation $C_yP\omega_n^{*}$ semigroups is a promising class of results that can be utilized to demonstrate interesting and practical results to connect discrete mathematics with real analysis, among other pure mathematical fields of study.

Lemma 5: Let $P\omega_n^{*0}$ be a finite star-like transformation semigroup adjoined with zero element (partial) $\emptyset$. Let 7 $f^{*} \in P\omega_n^{*0}$be a transformation with active non-zero domain $I(f^{*}) = Dom(f^{*}) \setminus \{0\}$. An element $f^{*} \in P\omega_n^{*0}$ belongs to the cyclicpoid class G_{cy}^{*} of order $m = |I(f^{*})| > 1$ if and only if f^{*} satisfies the following axioms:

(i) $f^{*}(0) = 0$ and $f^{*}(\zeta^{*}) \neq 0$ for all $\zeta * \in I(f^{*})$
(ii) $f^{*}(\zeta^{*}{}_1) = f^{*}(\zeta^{*}{}_2) \Longrightarrow \zeta^{*}{}_1 = \zeta^{*}{}_2$ for all $\zeta^{*}{}_1$, $\zeta^{*}{}_2 \in I(f^{*})$
(iii) $v(\zeta^{*}) = m$ for all $\zeta^{*} \in I(f^{*})$ where $v(\zeta^{*}) = min\{k \geq 1 : f^{k}(\zeta^{*}) = \zeta^{*}\}$.

Proof. Suppose f^{*} is a cyclicpoid generator of order m in G_{cy}^{*}. By definition of cyclicpoids in star-like semigroup, f^{*} acts as a single cyclic shift on the active non-zero transformation $I(f^{*})$ around the disk point. Set $I = \{0, \dots, n\}$, where 0 represent the adjoined star-like diskpoint such that $I(f^{*}) = Dom(f^{*}) \backslash \{0\}$ denote the active non-zero index set. Thus, $f^{*}(0) = 0$. Now, for each $\zeta^{*} \in I(f^{*})$, let $v(\zeta^{*}) = min\{k \geq 1 : f^{k}(\zeta^{*}) = \zeta^{*}\}$ denote the orbital period of ζ^{*}. This directly implies injectivity on $I(f^{*})$ implies $\zeta^{*} \in I(f^{*})$ belong to a single closed orbit of exact length $m = |I(f^{*})|$, with local orbit length at $v(\zeta^{*}) = m| < f^{*} > \cdot \zeta^{*}|$. $\Longleftarrow$ Conversely, assume $(i) - (iii)$ hold. $c_1(\zeta^{*}) = 1$ are exluded since $c_1(1) = c_1(2) = 1$ voilates injectivity for $|I(f^{*}) > 1|$. (iii) ensures that f^{*} does not break into the isolated star-like fixed points but instead cyclically permutes all in active state in $I(f^{*})$ with minimal period m. Therefore, f^{*} generates a cyclicpoid structure of order m over $I(f^{*}) \cup \{0\}$. This was illustrated in Figure 1:

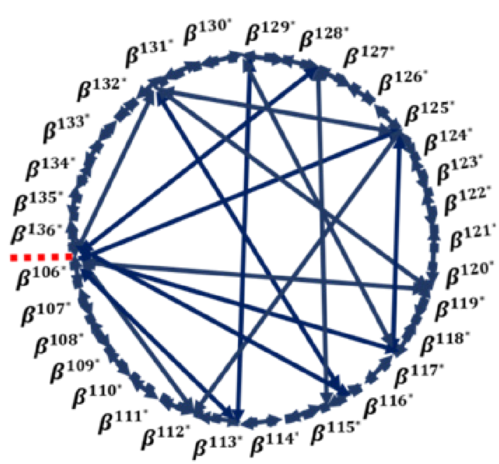

Fig. 1. Star-like cyclic group $(G_{cy}^{*} \ (C_c^{*}))$.

ACCEPTED MANUSCRIPT

This digraph represents the structural architecture of a cyclic cyclicpoid class where the group of f^* is strictly cyclic. The configuration is defined by the transformation of the set $a^* \setminus I(f^*)$ under the action of the star-like partial cyclic framework. It serves as a primary model for proving that in star-like cyclic structures, the kernel size is directly equivalent to the minimal length of the product decomposition of the transformation in to idempotents. The presence of fixed points stabilizes the orbit and reduces the maximal product length.

Proposition 1: Let $P\omega_n^{*0}$ be a finite star-like transformation semigroup over the max-plus semiring $(\mathbb{R}_{max}, \oplus, \otimes)$with adjoined zero element $0 = -\infty$ and $v = 0$. If $v \in \mathbb{R}^n$ is a star-like vector, then;

(i) $0 \otimes v = 0$

(ii) For any finite scalar $c \in \mathbb{R}$ and $v \in {\mathbb{R}^n}_{max}$, $c \otimes v = 0 \Leftrightarrow v = 0$

Proof.

(i) By definition of tropical scalar-vector multiplication, $(0 \otimes v)i = c \otimes v_i = -\infty + v_i = -\infty = 0$. For each $i \in \{1, \dots, n\}, c \otimes v = 0$.

(ii) If $v = 0$, applying (i) yield $c \otimes 0 = 0$. Assume $c \otimes v = 0$ for $c \in \mathbb{R}$. Evaluated component wise we have $c + v_i = -\infty$ for all $1 \leq i \leq n$. Since $c \in \mathbb{R}$ is a finite scalar, subtracting c from both sides gives $v_i = -\infty - c = -\infty = 0_i$. Therefore, $v = 0$.

Example 4: Let $V \in {\mathbb{R}^2}_{max}$, with adjoined zero vector $0 = \binom{-\infty}{-\infty}$. Consider $v = \binom{3}{1}$ .Then, scaling by $0 = -\infty$ give $0 \otimes v = -\infty \otimes \begin{pmatrix} -\infty & + & 3 \\ -\infty & + & 1 \end{pmatrix} = \binom{-\infty}{-\infty} = 0.$

Let $c = 2$, then, $c \otimes v = 2\binom{3}{1} = \begin{pmatrix} 2 & + & 3 \\ 2 & + & 1 \end{pmatrix} = \binom{5}{3} \neq 0.$

Using Lemma 5, and (ii) of Definition 4, we established that $c \otimes v$ equals 0 if and only if $v = 0$. Hence, finite max plus does not annihilate non-zero condition vectors.

Theorem1: Let $1 \leq p \leq n - 1$ and let the kernel operator $K_{\beta^*}(C_y P\omega_4^*) = 96$, satisfy the cardinal relation $K_{\beta^*}(C_y P\omega_4^*) = 96 = m(C_y P\omega_4^*) + p(G_{cy}^*) - 138 + 59 - 1 = 96$ within the cyclipoid class sytructure on $P\omega_n^*(m, p)$. Suppose the following hypotheses holds:

(H_1): $rank\ (\beta^*) \leq n - 1$, where β^* is a non-permutation transformation satisfy rank restriction.

(H_2): The star-like orbital adjacency relation R_0 on $X = [0]$ defined $i \in \{R_{0,j}\}$ if and only if $p, m \geq 0$: $i \backslash \beta^{*p} = j \backslash \beta^{*p}$ is taken under its reflexive, symmetric, and transitive star-like path connected closure R^*. (H_3): $\zeta^* \in C_y P\omega_{n-1}^*$ act as a rank preserving idempotent operator. Then, there exists $\beta^* \in P\omega_{n(m,\ p+1)}^*$ and $\zeta^* \in C_y P\omega_{n-1}^*$ such that each transformation in $P\omega_n^*\ (m, p - 1)$ can be expressed as a product of idempotent element in $C_y P\omega_{n-1}^*$ given as $\beta^* = \zeta^* \beta^*$.

Proof. Suppose $G_{cy}^* \subseteq C_y P\omega_n^*$. Let β^* be star-like transformation and $\zeta^* \in C_y P\omega_{n-1}^*$ be its idempotent decomposition factor over the finite domain $X = [n]$. Consider the star-like explicit functional transformations

$$\beta^* = \begin{pmatrix} W_1 & W_2 & W_3 & W_4 & \cdots & W_p \\ b_1 & b_2 & b_3 & b_4 & \cdots & b_p \end{pmatrix}, b_i \in [n],\ \zeta^* = \begin{pmatrix} w_1 \\ w_1' \\ w_1'' \\ w_1''' \end{pmatrix} \in C_y P\omega_{n-1}^*,$$

Evaluating the cardinal base capacity $m(C_y P\omega_4^*) = 38$ and cyclic subclass $p(G_{cy}^*) = 59$ yields: $K_{\beta^*}(C_y P\omega_4^*) = 96$, which anchors the operational transformation across the cyclicpoid categorization as illustrated in Figure 1. To establish that connected transformations in $\lceil\beta^* = (V, E)$ define true equivalence classes, then evaluate the algebraic closure of the orbit relation R_0 on $= [n]$, for any $i \in X$, $p = 0$, $m = 0$ gives $i\beta^* = i = i\beta^* \implies iR^*\ i$.Then, $iR^*\ j$ implies there are $p, m \geq 0$: $i\beta^{*p} = j\beta^{*m}$ which symmetrically yields $j\beta^{*m} = i\beta^{*p}$. Extending to the symmetric edge relation R_s gives $iR_s j \iff (i, j) \in E$. Thus, the transitive closure $R_T = \cup_{k=1}^{\infty} {R_s}^k$ guarantees that star-like path concatenation $P_1(i, j) \cdot P_2(j, k)$ establishes transitivity: $iR_T j$ and $jR^* k \Longrightarrow iR^* k$. By (H_2), the quotient space X/R partitions $\lceil\beta^*$into connected components bijectively matching G_{cy}^*: $|X\ /\ R\ | = p(G_{cy}^*) = 59$.

Under Hypothesis (H_1), rank $(\beta^*) \leq n - 1$, satisfying Howie's [14] condition for idempotent generation in singular transformation semigroup. Applying Hypothesis (H_3), we see that the idempotent decomposition factor is a rank preserving idempotent operator ${\zeta^*}^2 = \zeta^*$ acting on the sub class structure. Thus, star-like composition with ζ^* leaves the digraphs domain invariants, then rank $(\zeta^*\ \beta^*) = rank\ (\zeta^*) \leq n - 1$. The compositional power chains across transformation spaces are governed by

$$\beta^{*p} \leq K_{\beta^*} < G_{cy}^* \leq 10 \leq \beta^{*p} \leq \beta^5 \cdot \beta^5 \leq 1 \leq \beta^{52} \leq 2 \leq \beta^{*p}. \qquad (3)$$

Evaluating across the specific cyclicpoid sub classes yields:

ACCEPTED MANUSCRIPT

Tricyclic constraint class: $(\beta^3)^2 < 18$ as seen in Figure 2. This digraph displays the architecture of a tricyclic cyclicpoid class within the star-like transformation semigroup $C_yP\omega_n^*$. The category is characterized by Equation (1) where v^*denotes the number of star-like joints. The Figure 2 was used to visualizes the algebraic bound in Equation (5), establishing that for this category, the star-like period β^8 satisfies $(\beta^8)^2 \leq 18$. We used the Figure 2 to establish that even complex tricyclic transformations can be decomposed into a product of simpler star-like idempotent elements within the partial cyclic framework.

Polycyclic constraint class: $(\beta^4)^2 < 24$ as seen in Figure 3. This digraph illustrates the structural architecture of a polycyclic cyclicpoid class within the star-like transformation semigroup $C_yP\omega_n^*$. Figure 3 characterizes the quantity of star-like idempotent products derived from the relationship in Equation (3), where m and p represent the star-like index and period, respectively. The configuration demonstrates the spinnable nature of the class, bounded by Equation (3) and serves to visualize the equivalence between kernel size, product decomposition lengths, and transformation behavior in higher-order polycyclic structures. A visual representation of how the periodicity value of β^* interacts with the star-like index to define the group's boundaries is presented in Figure 3. It specifically models the polycyclic category products in G_{cy}^*, showing how these transformations can be expressed as a product of idempotents in $C_yP\omega_{n-1}^*$.

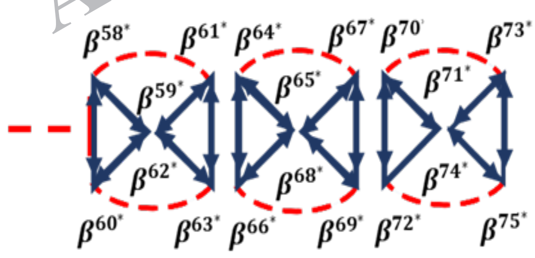

Fig. 2. Star-like tricyclic group $\left(G_{cy}^*(T_c^*)\right)$.

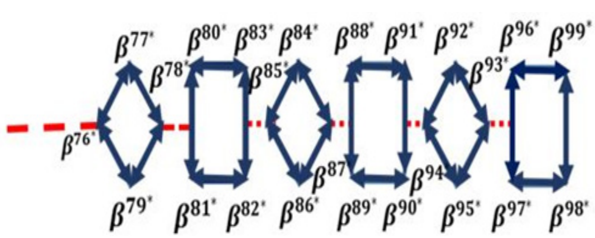

Fig. 3. Star-like polycyclic group $\left(G_{cy}^*(P_c^*)\right)$.

Acyclic constraint class: $(\beta^2)^2 < 6$ as seen in Figure 4. This digraph represents the structural configuration of an acyclic cyclicpoid class where the group of β^* is trivial. This contraint class is defined by the unique element x1 be the unique element in $a^*\backslash I(\beta^*)$, where $a^* = \{x_1^*, x_2^*, \dots x_a^*\}$. Figure 1, illustrates the idempotent product property in $C_yP\omega_{n-1}^*$, characterized by the numerical relationship in Equation (3). This visualization confirms the algebraic bound in Equation (4), establishing that the transformation is a product of idempotents within the star-like partial cyclic framework. For the acyclic category, we derive the specific bound $(\beta^5)^2 < 20$ was derived. Also, the Figure 1 aids in identifying the kernel K_{β^*}, which is defined as the set of elements within the cyclicpoid class that return to themselves after p iterations (where $p > 1$). This acyclic category serves as a base case for the more complex tricyclic and polycyclic structures, demonstrating how star-like cyclicpoid classes can be categorized based on their group properties. Because rank$(\beta^*) < n$, ζ^* satisfies sub chain invariance. By (H_3), star-like transformation guarantees that β^* factors into a product of idempotents $\zeta^* = \zeta^*{}_1 \ \dots \ \zeta^*{}_k \in C_yP\omega_{n-1}^*$, satisfying $\beta^* = \zeta^* \ \beta^*$. Consequently, every transformation in $P\omega_n^* \ (m, p-1)$ is expressible as a product of idempotent elements in $C_yP\omega_{n-1}^*$.

Remark 1: Let $m(C_yP\omega_4^*) = 34$, $p(G_{cy}^*) = 59$, yielding $K_{\beta^*} = 38 + 59 - 1 = 96$. For any $\beta^* \in C_yP\omega_4^*$ with rank$(\beta^*) = 3 \leq 3(H_1)$,taking the transitive path closure $R_T(H_2)$ partitions the orbits space into exactly $|X \,/\, R_T| = 59$ connected transformations. Applying the rank-preserving idempotents $\zeta^* \in C_yP\omega_3^* \ (H_3)$ under the tricyclic constraint $(\beta^*)^2 < 18$ completes the factor identity $\beta^* = \zeta^* \ \beta^*$.

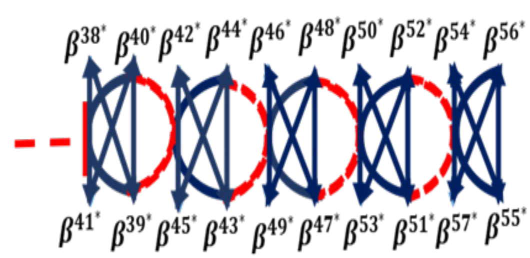

Fig. 4. Star-like acyclic group $\left(G_{cy}^*(A_c^*)\right)$.

ACCEPTED MANUSCRIPT

3.3 A Class of Spinnable Combinatorial Functions in $f(n, C_y P\omega_n^*)$ that are Cyclicpoid

This section presents the combinatorial characterization of the star-like sequences of G_{cy}^* in the star-like finite chain transformation semigroup. Building directly on the path-connected equivalence partition X/R established under hypothesis (H_2) of Theorem 1, we now evaluate the global cardinality of the resulting spinnable star-like transformation classes $S(P\omega_n^*)$ when the orbital components exhibit uniform cyclic periodicity p.

Theorem 2: Let X/R be the equivalence partition of $P\omega_n^*$ under Hypothesis (H_2) of Theorem 1, with base capacity $m = |X/R| - p + 1 \in \mathbb{Z}$ and cyclic class periodicity $p \in \mathbb{Z}$. If the structural parameter pair (m, p) satisfies the cyclic divisibility condition p|m2, then the cardinality of the spinnable class $S(P\omega_n^*)$ is a well-defined integer given by
$|S(P\omega_n^*)| = \frac{m^2+p}{p} = \frac{m^2}{p} + 1 \in \mathbb{Z}^* \cdot$

Proof. From Theorem 1, Hypothesis (H_2) guarantees that the equivalence closure R partitions the transformation orbit space into |X/R| disjoint connected components. Under the admissibility constraint $p \mid m^2$, the total number of non-trivial condition transitors across these components is divisible by the cyclic periodicity p. Thus, there exists a positive integer $k \geq 1$ such that $m^2 = kp$. Substituting $m^2 = kp$ into the cardinality expression yields: $|S(P\omega_n^*)| = \frac{kp+p}{p} = \frac{p(k+1)}{p} = k + 1$.

Since $k \in \mathbb{Z}+$, $k+1 \in \mathbb{Z}^+$. Therefore, $p|m^2$ ensures both algebraic consistency and integrality for the connectivity measure of $S(P\omega_n^*)$.

Example 5: Consider the semigroup $P\omega_4^*$ on $X^* = [4]$. Then, apply Theorem 2, using partition parameters established in Theorem 1 for $P\omega_4^*(X/R) = 5, p = 19$ such that

$m = |X/R| - p + 1 = 59 - 19 + 1 = 38$

Then, $m^2 = 382 = 1444$. Since, $1444\ 19 = 76$, $p|m^2$ holds with $K = 76$ Therefore, $|S(P\omega_n^*)| = \frac{1444}{19} + 1 = 76 + 1 = 77 \in \mathbb{Z}^+$.

Hence, the parameters inherited from Theorem1 automatically satisfy $p|m^2$ and yield a valid integer cardinality. For the cyclopoid transformation class $f(n, C_y P\omega_n^*)$, let $F_v: P\omega_n^* \to P\omega_{n-v}^*$ denote the canonical rank-reducing folding operator that contracts v defect elements onto the core equivalence closure R. Let $v^* : P\omega_n^* \to \mathbb{Z}^+$ be the joint valuation function mapping cyclopoid transformation orbits to their active-domain capacities, and let $G_{cy}(H_c) \leq \mathbb{Z}^+$denote the cyclic stabilizer subgroup acting on the core component $H_c \in$ X/R.

Lemma 6: Let $F_v: S(P\omega_n^*) \to S(P\omega_{n-v}^*)$ be the star-like folding operator defined by constructing v defect elements $D \subseteq X$ onto the equivalence closure R, where $|D| = v$ and $0 \leq$ v. Then F_v induces a well-defined bijection between $S_v(P\omega_n^*)$ and the product space $\binom{X/R}{v} \times O_{n-v}$, where O_{n-v} is the set of invariant orbit classes of size $n - v^*$.
Proof. Let $\alpha^*, \beta^* \in S_v(P\omega_n^*)$ be such that $F_v(\alpha^*) = F_v(\beta^*)$. Since F_v restricts star-like transformations to their non-singular active domain X\D while fixing contracted defect components identically on R, $F_v(\alpha^*) = F_v(\beta^*) \Rightarrow \alpha^*|_{X\setminus D} = \beta^*|_{X\setminus D}$ and $\alpha^*|_D = \beta^*|_D$

Thus $\alpha^* = \beta^*$, suppose there exists another star-like transformation γ^*being a target class $\gamma^* \in S(P\omega_{n-v}^*)$ and choice of v defect positions $D \in \binom{X/R}{v}$. Construct $\alpha^* \in PW_n^*$ by setting
$\alpha^*(x) = \gamma^*(x)\ \ (x \in X\setminus D), \alpha^*(x) = R(x)\ \ (x \in D).$

To show that $S_v(P\omega_n^*)$ is surjective, construct $\alpha^* \in S_v(P\omega_n^*)$, then $F_v(\alpha^*) = \gamma^*$ .Thus F_v is surjective.

Theorem 3: Let $P\omega_n^*$ be a star-like transformation semigroup satisfying hypothesis (H_2) of Theorem 1 and the cyclic divisibility condition $p|m^2$ of Theorem 2, where $m = |X/R| - p + 1$. For any defect function $v \in \mathbb{Z}$ in the admissible domain $0 \leq v \leq min\{n, m\}$, the cardinality of the v-folded cyclicpoid spinnable class $S_v(P\omega_n^*) \subset f(n, C_y P\omega_n^*)$ under the joint valuation v* is given by

$$|S_v(P\omega_n^*)| = \binom{v^*(P\omega_n^*)}{v} \left[\frac{|S(P\omega_n^*)||G_{cy}(H_c)|}{p(n-v)}\right],$$

where $|S(P\omega_n^*)| = \frac{m^2}{p} + 1$ is the total spinnable class cardinality established in Theorem 2, and $v^*(P\omega_n^*) = m^*$.

Proof. By Hypothesis (H_2) of Theorem 1, the equivalence closure R partitions the transformation semigroup $P\omega_n^*$ into disjoint equivalence components X/R, establishing a well-defined

ACCEPTED MANUSCRIPT

base capacity $m = X/R - p + 1$. Applying the joint valuation function v^* to $P\omega_n^*$ within the cyclicpoid framework $f(n, C_y P\omega_n^*)$ yields the effective capacity value $(P\omega_n^*)v^* = m$. Under the rank-reducing star-like folding operator F_v, as established in Lemma 6, precisely v defect functions are contracted onto the core equivalence closure R, leaving n−v active target conditions. By Lemma 6, the number of distinct choices for the v contracted star-like defect positions from the available capacity functions evaluated by $v^*(P\omega_n^*)$ is given by $\binom{v^*(P\omega_n^*)}{v} = \binom{m}{v}$. Because the parameter domain is strictly restricted to $0 \le v \le min\{n, m\}$, this factor is guaranteed to be a well-defined, non-zero positive integer. By Theorem 2, the total cardinality of the unfolded spinnable star-like class is given by $|S(P\omega_n^*)| = \frac{m^2}{p} + 1$, guaranteed to be an integer under the cyclic divisibility constraint $p|m^2$. Each core component $H_c \in X/R$ possesses local cyclic symmetries governed by the stabilizer subgroup $G_{cy}^*(H_c) \le Z_p$. Hence, the cyclopoid class is given by:

$$|S(P\omega_n^*)||G_{cy}^*(H_c)| \tag{4}$$

The folding operator F_v restricts Equation (2) across the $n - v$ remaining active non-singular target cores under a period p action. The resulting quotient $\frac{|S(P\omega_n^*)||G_{cy}^*(H_c)|}{p(n-v)}$, represents the exact weighted density of invariant cyclopoid orbits per active vertex. Because $v < n$, $n - v \ge 1$ is strictly positive, since, $|C_{cy}(H_c)| \ge 1$ and $|S(PN_n^*)| \ge 1$. The proof is complete.

Example 7: Consider

$$f(4, G_y P\omega_n^*) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & \phi \end{pmatrix}_i \in S(P\omega_n^*)$$

$n = 4, v < m = 38.0: 0 \le v = 1 < n.$

The active domain size is, $n - v = 4 - 1 = 3 \ne 0$. Then, $v^*(P\omega_4^*) = m = 38$. For the maximal cyclic action on the core H_c, $|G_{cy}(H_c)| = p = 19$. Therefore,

$$\binom{v^*(P\omega_4^*)}{v} = \binom{38}{1} = 38$$

To obtain the cardinality of the v^* star-like folded cyclopoid spinnable star-like class, substitute the values:

$$\left[\frac{|S(P\omega_4^*)||G_{cy}(H_c)|}{p(n-v)}\right] = \left[\frac{17 \times 19}{19(4-1)}\right] = \left[\frac{77}{3}\right] = 25.$$

Hence, $|S_v(P\omega_4^*)| = 38 \times 25 = 950$.

Lemma 7: Let $\beta_1^*, \beta_2^* \in f(n, C_y P\omega_n^*)$ be spinnable star-like transformations satisfying hypothesis (H_2) of Theorem1. If β_1 and β_2 share the same image rank $q = |Im(\beta_1^*)| = |Im(\beta_2^*)|$ and bonding index parameter $r = |I(\beta_1^*)| + 1 = I|I(\beta_2^*)| + 1$ then, their generated subsemigroups $\langle \beta_1^* \rangle$ and $\langle \beta_2^* \rangle$ are equivalently isomorphic as transformation semigroups.

Proof. Suppose $|I(\beta^*)| = r - 1$, the joint pair(q, r) gives a unique point structure $(q - r + 1,\ r - 1)$ consisting of $q - r + 1$ cyclic components and $r - 1$ bondingcomponents in X/R under hypothesis (H_2) of Theorem 1. $(\mathbb{Z}_p, \cdot)$ is transitive and uniform with local stabilizers $G_{cy}^*(H_c)$. Because the cycle length of the orbits is uniformly constrained by p and $G_{cy}^*(H_c)$, any two $\beta_1^*, \beta_2^* \in f(n, C_y P\omega_n^*)$ have same vector (q, r) of transformation semigroup isomorphism.

Theorem 4: Let $P\omega_n^*$ be a star-like transformation semigroup satisfying Theorems 1, 2, and 3, with base domain capacity $m = |X/R| - p + 1$ and cyclic periodicity p. For any spinnable transformation $\beta^* \in f(n, \zeta/P\omega_n^*)$ characterized by rank $q = |Im(\beta^*)|$ and boundary index $r = [I(\beta^*)] + 1$, for admissible domain $1 \le r \le q \le m$, the cardinality of $|S_{q,r}(P\omega_n^*)|$ is uniquely determined by the joint partition invariant vector (q, r) given by:

$$|S_{q,r}(P\omega_n^*)| = \binom{m}{q}\binom{q}{r-1}\left[\frac{|S(P\omega_n^*)|}{p(q-r+1)}\right],$$

where $|S(P\omega_n^*)| = \frac{m^2}{p} + 1$ is the total spinnable class established in Theorem 2.

Proof. By Lemma 7, the joint invariant vector (q, r) uniquely specifies the partition structure $(q - r + 1, r - 1)$ up to transformation semigroup isomorphism. Selecting q active image components out of the m available in X/R yields $\binom{m}{q}$ domain choices, with boundary structure $\binom{q}{r-1}$. By Theorem 1 (H_2), the remaining $q - r + 1$ act as cyclic under p. From Theorem 2, $|S(P\omega_n^*)|$ yields $\frac{|S(P\omega_n^*)|}{p(q-r+1)}$.

Since $1 \le r \le q \le m$, the denominator satisfies $q - r + 1 \ge 1$, so the quotient is well-defined. Hence the stated cardinality formula follows.

ACCEPTED MANUSCRIPT

Example 8: Consider $\beta^* = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & \phi \end{pmatrix} \in P\omega_4^*$ with $m = 38$, $= 19$, $|S(P\omega_n^*)| = 77$. Evaluate the rank $q = 3|Im(\beta^*)|$ and $r = 2|I(\beta^*)| = 1$. Then, we see that

$1 \le r = 2 \le q = 3 \le m \le 38,$

$q - r + 1 = 3 - 2 + 1 = 2 \ge 1.$

Indicating two cyclic and one boundary. Thus, $\binom{m}{q} = \binom{38}{3} = \frac{38\times37\times36}{3+2+1} = 8436,$ and $\binom{q}{r-1} = \binom{3}{1} = 3$, gives:

$$\left[\frac{|S(PN_4^*)|}{p(q-r+1)}\right] = \left[\frac{77}{19\times2}\right] = \left[\frac{77}{38}\right] = 2$$

Therefore, $|S_{3,2}(P\omega_4^*)| = 50\,616$.

Theorem 5: Let $\beta^* \in C_yP\omega_n^*$ be a spinnable transformation within a star-like cyclicpoid class over the classified digraph taxonomy $G \in \{A_c, T_c, H_c\}$ representing Acyclic, Tricyclic, and Hexcyclic, categories, respectively. Assume $C_yP\omega_n^*$ satisfies the centralizer action axiom: for all star-like transformations $\varphi_i^* \in C_yP\omega_n^*$, and structural generators $\tau \in G_{cy}^*, \varphi^* \cdot \tau = \tau \cdot \varphi^*$ Then:
(i) The generated subsemigoup $\langle C_yP\omega_n^*\rangle_i$ under map composition is commutative.
(ii) The spinnable class operator ΔG_{cy}^* satisfies $m(C_yP\omega_n^*) + p(C_yP\omega_n^*) \le \Delta G_{cy}^*$, evaluated across digraph structural classes via lower and upper combinatorial functions $f(n, C_yP\omega_n^*)$ and $f(n, G_{cy}^*)$.

Proof. Suppose every star-like transformation ${\varphi_i}^* \in C_yP\omega_n^*$ belongs to the centralizer of the generator set $G_{cy}^* : {\varphi_i}^* \in C_yP\omega_n^* (G_{cy}^*) \Leftrightarrow {\varphi_i}^* \cdot \tau = \tau \cdot {\varphi_i}^*$.

By definition, is a finite composition of the structural generators $\tau_1, \tau_2, \dots, \tau_k \in G_{cy}^* : \beta^* = \tau_1, \tau_2, \dots, \tau_k$. Then $\varphi^* \cdot \tau = \tau \cdot \varphi^*$, for all ${\varphi_i}^* \in C_yP\omega_n^*$ and $\tau \in G_{cy}^*$ satisfies $(\varphi^* \cdot \beta^*)(x) = \varphi^*(\beta^*(x)) = \beta^*(\varphi^*(x)) = (\beta^* \cdot \varphi^*)(x)$ for all $\varphi^*, \beta^* \in C_yP\omega_n^*$ and domain element x. This shows abelianity features of star-like semigoup $C_yP\omega_n^*$ under map composition. Evaluated across the structural digraph classes, with cycle order $c = 0,3,6$ and $c \le 5$, the lower operational combinatorial function is established as:

$$f(n, C_yP\omega_n^*) = \begin{cases} |K_{\beta^*}(G_{cy}^*)| \le K_{\beta^*} Cycl\,(i\beta^*); i \ge 3, \\ |m(G_{cy}^*) + p(G_{cy}^*) - 1| \le \left(G_{cy}^*(H_c^*)\right); m \ge p \ge 4, \\ |K_{\beta^*}(Cycl\,(i\beta^*))| \le G_{cy}^*(P_c^*) + G_{cy}^*(H_c^*); c \ge 5, \end{cases} \tag{5}$$

where $Cycl\,(i\beta^*)$ counts structural cycles. To establish the upper operational combinatorial function, we apply the orbit-stablizer Theorem to the non-fixed state space $V \backslash fix(\beta^*)$, which has cardinality $|V \backslash fix(\beta^*)| n - fix(\beta^*)$. Because the centralizer $Z(\beta^*)$ partitions this non-fixed state space into uniform $p-$ periodic transformations in the H_c^* class, we see that $p \cdot G_{cy}^*(H_c^*) = n + Cycl\,(i\beta^*) - fix(\beta^*)$. Lemma 8 and Equation (5) yields the piece wise operational upper combinatorial function:

$$f(n, G_{cy}^*) = \begin{cases} \binom{m}{p} (C_yP\omega_n^*) \le G_{cy}^*(P_c^*); m \ge p \ge 3, \\ \binom{m+1}{p-1} + \sum_{r^*=1}^{c} z_{r^*}^* + T_c^* \le p\left(G_{cy}^*(H_c^*)\right); m \ge p \ge 1, \\ \binom{m+1}{p-1} + \sum_{r^*=1}^{c} z_{r^*}^* + H_c^* \le n + Cycl\,\beta^* - fix\,\beta^*; H_c^* \ge r^* \ge n \ge 3, \end{cases} \tag{6}$$

Summation and index aggregation across m by $p(1 \le p \le m)$ constraint the operational capacity of each components: $\sum_{p=1}^{m} f(n, G_{cy}^*) \le \Delta G_{cy}^*(\beta^*)$. Separating the component-dependent $(G_{cy}^* - 1)$ form the centralizer invariant base weight, we have:

$$\sum_{p=1}^{m} \left[(G_{cy}^* - 1) + C_yP\omega_n^*\right] \le \Delta G_{cy}^*(\beta^*),$$

such that:

$$\sum_{p=1}^{m} (G_{cy}^* - 1) + \sum_{p=1}^{m} C_yP\omega_n^* \le \Delta G_{cy}^*(\beta^*).$$

Then, $(C_yP\omega_n^*) + p(G_{cy}^* - 1) \le \Delta G_{cy}^*$. By Theorem 4, every spinnable generator $\beta^* \in C_yP\omega_n^*$ uniquely characterizes the class representatives $G_{cy}^*(\beta^*)$ equal to its minimal relapse index: $G_{cy}^*(\beta^*) = r^*(\beta^*)$ yields:

$$G_{cy}^*(\beta^*) \cap C_yP\omega_n^* = \bigcup_{n=1}^{\infty} \left(G_{cy}^*\left(\beta^* + \frac{n^2}{r^*}\right)\right)\left(2G_{cy}^*\left(\beta^* + \frac{r^*}{3n}\right)\right),$$

where $\frac{n^2}{r^*}$ count the pair combinatorics across level and the linear term $\frac{r^*}{3n}$ normalizes cycle-density across sub-kernels. Therefore, using the combinatorial collapse-relapse constraint functions in Equation (6) generate:

$$(n, G_{cy}^*, r^*, q^*) = \binom{2n - q^* + 1}{(2n - q^*) - (G_{cy}^*(\beta^*) - r^*)} = \binom{2n - q^* + 1}{2n - G_{cy}^*(\beta^*) + r^* - q^*}.$$

which shows that the operational capacity bound ΔG_{cy}^* holds generally across all digraph classes.

Theorem 6: Let $\beta^* \in C_yP\omega_n^*$ be a spinnable transformation semigroup over $G \in \{A_c, T_c, H_c\}$ with collapse index $q^*(\beta^*)$ and relapse index $r^*(\beta^*)$ satisfying that $r^*(\beta^*) \le q^*(\beta^*)$. Under

ACCEPTED MANUSCRIPT

the centralizer action established in Theorem 5 and given initial boundary conditions $f_{0,G^*_{cy}} = 0$ and $f_{0,G^*_{cy}} = G^*_{cy}(\beta^*) - 1$. the lower bound combinatorial function $f_{n,G^*_{cy}}$ satisfies the second-difference tate transition identity:
$f_{n+1\,G^*_{cy}} - 2f_{n\,G^*_{cy}} + f_{n-1\,G^*_{cy}} = r^*(\beta^*)\, p \cdot r^*(\beta^*)$
for $\frac{q^*(\beta^*)}{m} \geq \frac{r(\beta^*)}{p}$, then:

$$f_{n,G^*_{cy}}(\beta^*, \quad q^*, \quad r^*, \quad m, \quad p) = \left(\frac{q^*(\beta^*)+r^*(\beta^*)}{2G^*_{cy}(\beta^*)} - 1\right) + \left(\frac{G^*_{cy}(\beta^*) - n\cdot m}{p\cdot q^*\ (\beta^*)}\right).$$

Proof. By Definition 7, the domain X_n partitions into cyclic components $Ccyl(G^*_{cy}(\beta^*))$ and defect states $X_n \backslash Im(\beta^*)$, given $m = Im(\beta^*) + q^*(\beta^*)$. The star-like paths across $q^*(\beta^*)$ satisfying $(n - q^*(\beta^*))2 + 2r^*(\beta^*)$. Linking the joint partition invariant vector (q,r) of Theorem 4 directly to the boundary structure. By Theorem 5, centralizer commutativity $\varphi^* \cdot \tau = \tau \cdot \varphi^*$ ensures uniform operator action ΔG^*_{cy} across paths with increment $\Delta f_n = f_{n+1} - f_n + \frac{r^*(\beta^*)}{p\cdot q^*\ (\beta^*)} \cdot n.$

Taking difference yields $\Delta^2 f_n = (f_{n+1} - f_n) - (f_n - f_{n+1}) = \frac{r^*(\beta^*)}{p\cdot q^*\ (\beta^*)}$ such that:

$$2A = \frac{r^*(\beta^*)}{p\cdot q^*\ (\beta^*)} \Longrightarrow A = \frac{r^*(\beta^*)}{2p\cdot q^*\ (\beta^*)},$$

$$f_0 = 0 \Longrightarrow C = 0.$$

Then, $f_1 = A(1)^2 + B(1) = \frac{r^*(\beta^*)}{2p\cdot q^*\ (\beta^*)} + B = G^*_{cy}(\beta^*) - 1.$

Solving, $= G^*_{cy}(\beta^*) - 1 - \frac{r^*(\beta^*)}{2p\cdot q^*\ (\beta^*)}$. Susbstituitng $A, B,$ and C into $f_n = A_n{}^2 + B_n, +C$ yields $f_{n,G^*_{cy}} = (\frac{r^*(\beta^*)}{2p\cdot q^*\ (\beta^*)})n^2 + (G^*_{cy}(\beta^*) - 1 - \frac{r^*(\beta^*)}{2p\cdot q^*\ (\beta^*)})n.$

From Lemma 6, $G^*_{cy}(\beta^*) - v^*_1(\frac{q^*(\beta^*)}{m})) + v^*_2(\frac{r^*(\beta^*)}{p})$, then, by Definition 1, Equation (6) shows that $\frac{q^*(\beta^*)}{m} \geq \frac{r^*(\beta^*)}{p}$ that established $f_{n,G^*_{cy}}(\beta *, q *, r *, m, p) = \frac{q^*(\beta^*)+r^*(\beta^*)}{2G^*_{cy}(\beta^*)} + (G^*_{cy}(\beta^*) - \frac{n\cdot m}{p\cdot q^*\ (\beta^*)})\cdot$

Hence, the proof.

4. DISCUSSION AND CONCLUSION

4.1. Comparative Analysis with Existing Literature

To contextualize the theoretical contributions of the proposed star-like cyclicpoid framework $C_yP\omega^*_n$, we compare its structural capabilities with foundational and recent transformation semigroup models in Table 3.

Unlike traditional monogenic semigroups where elements outside the kernel operate strictly as linear tails feeding into a cyclic group subsemigroup, the star-like cyclicpoid accommodates partial unary maps with zero-adjunctions $(\emptyset)$. Compared to recent restricted equivalence models [23, 24], our framework introduces tropical matrix spectrum evaluation via min-plus algebra, bridging abstract transformation theory with spectral graph methods and operational path counting.

4.2. Mathematical Implications

The mathematical formulation presented in Section 3 carries key theoretical implications for algebraic semigroup theory:

(i) Spectral Resolution of Non-Group Kernels: Mapping non-group transformation matrices onto

ACCEPTED MANUSCRIPT

Table 3. Comparative summary of transformation semigroup frameworks.

Semigroup framework	Non-Group kernel support	Unary inversion duality	Tropical spectral representation	Primary application domain
Classical monogenic (Howie [14])	Limited to $K_{\beta^*} \cong C_p$	No	No	Abstract algebra
Order preserving isometries (Ali *et al.* [12])	Partial	No	No	Combinatorics on Finite chains
Equivalence retricted (Srimora *et al.* [23])	Structural	No	No	Invariant set theory
Star-like cyclicpoid $(C_yP\omega^*_n)$. This work	Full support	Explicit (G^*_{cy})	Min-plus polynomials	Algebraic and tropical systems

tropical minplus polynomials allows the extraction of invariant tropical roots α_i^*. This proves that min-plus linear algebra can effectively bypass the traditional lack of multiplicative inverses in partial transformation semigroups.

(ii) Tight Parameter-Driven Lower Bounds: The combinatorial lower bound formulation established in Theorem 3 links generator indices m, periods p , and joint parameters v^*. This provides a predictive algebraic tool for evaluating valid transformation paths without relying on full Cayley table enumeration.

4.3. Practical Applications

Beyond pure semigroup theory, the proposed framework offers concrete utility across several applied mathematical domains:

(i) Biological Regulatory Network Modeling: Star-like partial transformations naturally model biological networks where regulatory pathways exhibit threshold-triggered cyclic feedback loops (kernel period p) initiated by non-reversible startup conditions (index m).

(ii) Deterministic Finite Automata (DFA) Optimization: In computer science, unary state transitions containing undefined partial states $(\emptyset)$ can be evaluated using the derived tropical root bounds to optimize transition path routing and state-space complexity.

(iii) Tropical Bottleneck Pathfinding: The min-plus polynomial matrix framework provides an explicit algebraic foundation for solving dynamic programming and bottleneck shortest-path problems in weighted networks.

4.4. Concluding Remarks

The mathematical framework developed across these cyclicpoid classes and star-like transformation semigroups bridges classical semigroup theory, tropical matrix algebra, and structural graph dynamics. By establishing explicit operational bounds and formalizing the interplay between idempotent commutativity and tropical characteristic roots, these results offer a unified algebraic foundation for analyzing complex finite state systems, routing optimization, and generalized inverse structures.

5. CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

6. ETHICAL STATEMENT

No ethics approval was required for this study.

7. FUNDING

No funding was received to conduct this research.

8. AUTHORSHIP CONTRIBUTION STATEMENT

All authors equally contributed in this study.

9. DECLARATION

The authors declare that: (i) the results presented in this manuscript are original; (ii) the manuscript has not been published previously and is not under consideration for publication elsewhere; (iii) all authors have read and approved the final manuscript; and (iv) in the event of acceptance, copyright will be assigned to the Pakistan Academy of Sciences. All permissions for use of copyrighted material from other sources have been obtained.

ACCEPTED MANUSCRIPT

10. REFERENCES

1. D. Maclagan and B. Sturmfels (Eds.). Graduate Studies in Mathematics: Introduction to tropical geometry. *American Mathematical Society, Providence, Rhode Island, USA* (2015). https://doi.org/10.1090/gsm/161
2. I. Itenberg, G. Mikhalkin and E. Shustin (Eds.). Tropical Algebraic Geometry (Second Edition). *Birkhäuser Basel, Switzerland* (2009). https://doi.org/10.1007/978-3-0346-0048-4
3. P.A. Helminck and Y. Ren. Generic root counts and flatness in tropical geometry. *Journal of the London Mathematical Society* 111(5): e70171 (2025). https://doi.org/10.1112/jlms.70171
4. A.J. Puentes, H. Markwig, S. Pauli, and F. Röhrle. Quadratically enriched plane curve counting via tropical geometry. *arXiv preprint arXiv*: 2502.02569 (2025). https://doi.org/10.48550/arXiv.2502.02569
5. G.R. Ibrahim, G.N. Bakare, I.F. Usamot, and T.O. Aliu. Tropical geometric approach to signed full transformation semigroups. *The Transactions of the Nigerian Association of Mathematical Physics* 21: 205-212 (2025). https://doi.org/10.60787/tnamp.

v21.512
6. E. Katz. Lifting tropical curves in space and linear systems on graphs. *Advances in Mathematics* 230(3): 853-875 (2012). https://doi.org/10.1016/j.aim.2012.03.017
7. E. Brugallé and K. Shaw. A bit of tropical geometry. *The American Mathematical Monthly* 121(7): 563-589 (2014). https://doi.org/10.4169/amer.math.monthly.121.07.563
8. K. Shaw. A tropical intersection product in matroidal fans. *SIAM Journal on Discrete Mathematics* 27(1): 459491 (2013). https://doi.org/10.1137/110850141
9. I.F. Usamot, G.N. Bakare, and B.M. Ahmed. Tropicalization of idempotent elements of full transformation semigroups. *Confluence Journal of Pure and Applied Sciences* 2(1): 56-61 (2018). http://cjpas.org.ng/index.php/pub/article/view/33
10. U.F. Idayat, I.G. Risqot, B.G. Naimat, A.A. Dauda, and T.O. Aliu. Green's equivalence classes in full transformation semigroups: A height-based approach. Euler: *Jurnal Ilmiah Matematika, Sains dan Teknologi* 13(1): 45-49 (2025). https://doi.org/10.37905/euler.v13i1.30957
11. G.R. Ibrahim, G.N. Bakare, I.F. Usamot, O.I. Sulyman, S.A. Akinwunmi, and A.D. Adeshola. Tropical roots and their multiplicities on the subsemigroup of order-decreasing and order-preserving full transformation. *Lasu Journal of Research and Review in Science* 11: 121-133 (2024). https://kwasuspace.kwasu.edu.ng/handle/123456789/6113
12. B. Ali, M.A. Jada, and M.M. Zubairu. On the ranks of certain semigroups of order preserving partial isometries of a finite chain. *Journal of Algebra and Related Topics* 6(2): 15-33 (2018). http://doi.org/10.22124/jart.2019.10875.1109
13. G.U. Garba. On the idempotent ranks of certain semigroups of order-preserving transformation. *Glasgow Mathematical Journal* 36(1): 1-9 (1994). http://doi.org/10.1017/S0017089500030482
14. J.M. Howie. Fundamentals of semigroup theory. *Semigroup Forum* 58: 156-158 (1999). https://doi.org/10.1007/s002339900006
15. O. Ganyushkin and V. Mazorchuk (Eds.). Classical finite transformation semigroups: An introduction. *Springer-Verlag, London* (2009). https://doi.org/10.1007/978-1-84800-281-4
16. P.M. Higgins. Combinatorial results for semigroups of order-preserving mappings. *Mathematical Proceedings of the Cambridge Philosophical Society* 113(2): 281-296 (1993). https://doi.org/10.1017/s0305004100075964
17. S.A. Akinwunmi. The cyclicpoid life stages of a star-like mathematician: The cyclicpoid life stages. *Journal of the Nigerian Mathematical Society* 44(1): 139-151 (2025). http://ojs.ictp.it/jnms/index.php/jnms/article/view/1126
18. S.A. Akinwunmi, M.M. Mogbonju, and A.O. Adeniji. Multiplicative invertibility characterizations on star-like cyclicpoid $C_yP\omega_n^*$ finite partial transformation semigroups. *Hikari Journal of Pure Mathematical Science* 10(1): 45-55 (2021). https://doi.org/10.12988/pms.2021.91276
19. S.A. Akinwunmi, G.R. Ibrahim, A.O. Adeniji, and D.O. Oyewola. On the multiplicity order of spinnable star-like transformation semigroup. *Journal of the Nigerian Society of Physical Sciences* 6(1): 1692 (2024). https://doi.org/10.46481/jnsps.2024.1692
20. A.T. Imam and M.J. Ibrahim. On products of 3-paths in finite full transformation semigroups. *Algebra and Discrete Mathematics* 33(2): 60-77 (2022). https://doi.org/10.12958/adm1770
21. A. Umar. Some combinatorial results in the theory of partial transformation semigroups. *Algebra and Discrete Mathematics* 17(1): 110-134 (2014). http://admjournal.luguniv.edu.ua/index.php/adm/article/view/1027/551
22. A. Kachalla and B.A. Madu. Expressing partial order-preserving transformations as products of nilpotent. *International Journal of Development Mathematics* 1(3): 112-125 (2024). https://doi.org/10.62054/ijdm/0103.08
23. P. Srimora, R. Chinram and W. Yonthanthum. Regularity of the transformation semigroups with invariant sets and restricted by an equivalence relation. *Gulf Journal of Mathematics* 20(2): 194-202 (2025). https://doi.org/10.56947/gjom.v20i2.3191
24. R. Chinram, P. Petchkaew, S. Baupradist. Left and right magnifying elements in generalized semigroups of transformations by using partitions of a set. *European Journal of Pure and Applied Mathematics* 11(3): 580-588 (2018). https://doi.org/10.29020/nybg.ejpam.v11i3.3260
25. A. H. Clifford and G. B. Preston (Eds.). Mathematical Surveys: The Algebraic Theory of Semigroups. *American Mathematical Society, Providence, Rhode Island, USA* (1961). https://doi.org/10.1090/surv/007.1
26. S. Lipscomb (Ed.). Mathematical Surveys and Monographs: Symmetric Inverse Semigroups. *American Mathematical Society, Providence, Rhode Island, USA* (1996). https://doi.org/10.1090/surv/046

ACCEPTED MANUSCRIPT