Proceedings of the Pakistan Academy of Sciences: A Physical and Computational Sciences 60(2): 37-43
ISSN (Print): 2518-4245; ISSN (Online): 2518-4253
https://doi.org/10.53560/PPASA(60-2)813

Pakistan Academy of Sciences

Research Article

A Stable Version of the Modified Algorithm for Error Minimization in Combined Numerical Integration

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Abstract: The present study derives a stable version of "A Modified Algorithm for Reduction of Error in Combined Numerical Integration". It is discovered that the earlier proposed scheme "A Modified Algorithm for Error Reduction in Combined Numerical Integration" exhibits accuracy fluctuations when the number of slits, $\boldsymbol{n}$, is increased ($\boldsymbol{n \geq 9}$). . Starting with the number of slits $\boldsymbol{n = 9}$ and increasing the count of sub-intervals, the error increases spontaneously. This spontaneous spike in error is resolved by considering a better combination of quadrature rules. To this, the notable result of this study is the identification of an optimal choice for quadrature formulae that could minimizes error fluctuations in combined numerical integration regardless the number of slits ($\boldsymbol{n}$). With this revised choice, the error remains relatively stable and predictable even as the count of sub-intervals is increased.

Keywords: Numerical Integration, Weddle's Rule, Boole's Rule, Six-Point Rule, Simpson's 1/3 Rule.

1. INTRODUCTION

Quadrature is a mathematical technique that involves the computation of definite integrals. It has a plethora of applications in areas of engineering, finance, physics and beyound. In addition to its use in numerical integration, quadrature also has applications in the differential equations' solutions, such as in the finding solution to boundary value problems numerically. It is also used in Monte Carlo simulation, which is a statistical method for approximating the value of an integral by sampling from a probability distribution. Quadrature methods aim to approximate definite integrals with a desired accuracy. Throughout the past years, various quadrature rules and formulas have been devised to enhance the accuracy of this approximation.

Among the earliest quadrature rules is the trapezoidal rule, which involves estimating the area under a curve by summing the areas of trapezoids. Though it is simple, this method isn't always accurate, especially for highly curved functions that have jagged peeks and valleys. To remedy this limitation, new quadrature rules and formulae were created. The Newton-Cotes (NC) formula is one of the most commonly used quadrature formulae. This formula includes splitting the area under a curve into smaller subintervals and then using a polynomial to approximate the curve inside each subinterval. The obtained approximation is then utilised to calculate the integral. The accuracy of the NC formula depends on the degree of the polynomial and the number of subintervals used.

To approximate definite integrals, quadrature formulas are used frequently. Definite integrals that cannot be integrated analytically can be approximated by quadrature formulas [1]. Quadrature methods are an effective way of approximating integrals when the integrand's discontinuous behaviour is in a bounded range rather than the closed-form [2]. Newton–Cotes quadrature formulae are based on equally spaced points (abscissas) [3]. In mathematics, numerical integration (NI) is among the most basic and significant practices. It has a wide range of uses, including engineering, mechanics and physics. The primary purpose is to have an alternative mechanism for estimating given definite integrals within finite

Received: April 2023; Accepted: June 2023
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integration limits. Difficulties in NI can be traced back to Greek antiquity. They increased the number of sides of an inscribed polygon to calculate the area of a circle. With the development of calculus in the 17th century, new mathematics evolved, contributing to elementary rules in NI. Later, NI got more practical with the advent of computers, and at the day, we have numerous classical and modern algorithms providing speedy and more accurate results [4]. Many researchers and experts have already conducted substantial research on NI. A NI rule was proposed by Amanat [5], which is based on commonly used Quadrature methods like the Trapezoidal, Simpson's, and Weddle's rules. Natarajan *et al.* [6] explored the superconvergence of the NC rule for Cauchy principal value integrals while Liu *et al.* [7] compared various NI rules for approximating these integrals. A mid-point integration rule for nonlinear differential equations was proposed by Soomro *et al.* [8], and Shaikh *et al.* [9] proved the quadrature methods outperform polynomial collocation in solving second-kind Fredholm integral problems.

Bhatti *et al.* [10] presented a modified approach for error reduction in combined numerical integration (CNI) by combining lower-order rules for decreased error and enhanced accuracy. This method outperforms the original quadrature rules by two orders of magnitude. In a particular quadrature rule, the number of slits is a parameter for obtaining higher accuracy. It is supposed that number of slits give a better approximation of the integral than number of slits, where . The algorithm modified by Bhatti seems to work fine. However, the error in the approximation of the integral does not always follow a downward trend. It is observed that for number of slits, the error is sometimes higher than for number of slits, where , and this happens periodically. This is what we have tried to rectify in the present study.

2. METHODOLOGY USED FOR QUADRATURE RULES

2.1 The Newton-Cotes Formulae

The NC formulae represent a set of numerical integration techniques widely employed in approximating definite integrals of functions. These techniques are based on the concept of approximating a curve by a series of straight line segments These formulae offer a practical approach to solving integrals when an analytical solution is not easily attainable or simply doesn't exists. They are particularly useful for functions that are difficult to integrate analytically.

The family of NC formulae is a simple yet effective set of NI algorithms. Divide a function $f(x)$ across some interval $[a, b]$ into n equal pieces so that $f_n = f(x_n)$ and $h \approx \frac{b-a}{n}$. Then, identify polynomials that resemble the listed function and integrate them to get an idea of the area under the curve. It would be ok to use Lagrange's interpolation to pick the ideal polynomials. In this manner, the NC formulae or quadrature formulae are derived by Beyer [11].

NC formulas is considered open, or closed if the interval used is $[x_2, x_{n-1}]$ or $[x_1, x_n]$ respective. When the function is specified explicitly rather than tabulated against the values of x, the optimal NI approach is known as Gaussian quadrature. This approach yields more accurate approximations (but is substantially more difficult to execute) by selecting the intervals to sample the function by Hildebrand [12].

Following are the most commonly used formula. The trapezoidal rule refers to the 2-point closed NC formula because it approximates the integral by placing trapezoid(s) with a base parallel to the x-axis and an inclined top (linking the endpoints of the interval). Let x_1 and $x_2 = x_1 + h$ be the first the other endpoint then the Lagrange interpolating polynomial through the points (x_1, f_1) and (x_2, f_2) is:

$$\begin{aligned} P_2(x) &= \frac{x - x_2}{x_1 - x_2} f_1 + \frac{x - x_1}{x_2 - x_1} f_2 \\ &= \frac{x - x_1 - h}{-h} f_1 + \frac{x - x_1}{h} f_2 \\ &= \frac{x}{h}(f_2 - f_1) + (f_1 + \frac{x_1}{h} f_1 - \frac{x_1}{h} f_2) \end{aligned}$$

Upon integration throughout the interval, which corresponds to the area of the trapezoid, the result is:

$$\begin{aligned}\int_{x_1}^{x_2} f(x)\,dx &= \int_{x_1}^{x_{1+h}} P_2(x)\,dx \\ &= \frac{1}{2h}(f_2 - f_1)[x^2]_{x_1}^{x_2} \\ &\quad + \left(f_1 + \frac{x_1}{h}f_1 - \frac{x_1}{h}f_2\right)[x^2]_{x_1}^{x_2} \\ &= \frac{1}{2h}(f_2 - f_1)(x_2 + x_1)(x_2 - x_1) \\ &\quad + (x_2 - x_1)(f_1 + \frac{x_1}{h}f_1 - \frac{x_1}{h}f_2 \\ &= \frac{1}{2}(f_2 - f_1)(2x_1 + h) + f_1 h \\ &\quad + x_1(f_1 - f_2) \\ &= x_1(f_2 - f_1) + \frac{1}{2}h(f_2 - f_1) + hf_1 \\ &\quad - x_1(f_2 - f_1) \\ &= \frac{1}{2}h(f_1 + f_2) \\ &\quad - \frac{1}{12}h^3 f''(\zeta) \qquad (1)\end{aligned}$$

which is the trapezoidal rule. The final term indicates the margin of error, which is limited by the fact that $x_1 \le \zeta \le x_2$ cannot exceed the maximum value of $f''(\zeta)$ within this range.

The rule with 3 points is known as the Simpson's rule.

$$\begin{aligned}\int_{x_1}^{x_4} f(x)\,dx \\ &= \frac{3}{8}h(f_1 + 3f_2 + 3f_3 + f_4) \\ &- \frac{3}{80}h^5 f^{(4)}(\zeta) \qquad (3)\end{aligned}$$

The Boole's rule is a closed 5-point rule.

$$\begin{aligned}\int_{x_1}^{x_5} f(x)\,dx &= \frac{2}{45}h(7f_1 + 32f_2 + 12f_3 + 32f_4 \\ &\quad + 7f_5) \\ &\quad - \frac{8}{945}h^7 f^{(6)}(\zeta) \qquad (4)\end{aligned}$$

Higher order rules include the 6-point.

$$\begin{aligned}\int_{x_1}^{x_6} f(x)dx \\ &= \frac{5}{288}h\begin{pmatrix}19f_1 + 75f_2 + 50f_3 + 50f_4 \\ +75f_5 + 19f_6\end{pmatrix} \\ &- \frac{275}{12096}h^7 f^{(6)}(\zeta) \qquad (5)\end{aligned}$$

And the Weddle's rule

$$\begin{aligned}\int_{x_1}^{x_7} f(x)\,dx \\ &= \frac{h}{140}\begin{pmatrix}41f_1 + 216f_2 + 27f_3 + 272f_4 \\ +27f_5 + 216f_6 + 41f_7\end{pmatrix} \\ &- \frac{9}{1400}h^9 f^{(8)}(\zeta) \qquad (6)\end{aligned}$$

Generally, the n-point rule can be expressed analytically as:

$$\begin{aligned}\int_{x_1}^{x_n} f(x)\,dx \\ &= h\sum_{i=1}^{n} H_{n,i} f_i \qquad (7)\end{aligned}$$

Where,

$$\begin{aligned}H_{n,r+1} = \frac{(-1)^{n-r}}{r!\,(n-r)!}\int_0^n & t(t-1)\ldots(t-r+1)(t \\ & - r - 1)\ldots(t \\ & - n)\,dt \qquad (8)\end{aligned}$$

Note that,

$$\begin{aligned}\sum_{r=0}^{n} H_{n,r+1} \\ &= n \qquad (9)\end{aligned}$$

2.2 Single and Multiple Integration Rules

By dividing intervals into smaller parts and applying the technique to each segment, we can enhance the accuracy of the mentioned rules. These resulting equations are referred to as multiple or composite rules, Burden *et al.* [13].

The observed order of accuracy for the quadrature formulas is: Simpson's $\frac{3}{8}$ formula > Simpson's $\frac{1}{3}$ formula > Boole's formula > Trapezoidal formula > Weddle's formula by Amjad *et al.* [14].

2.3 Modified Algorithm for Combined Quadrature

The scheme by Amanat [5] uses the following method Trapezoidal, Simpsons $\mathbf{3/8}$, Boole's and Weddle's rule interchangeably. The number of sub-intervals in the methods is suggested to be greater than or equal to 9. Out of the total number of subintervals, the first 6 subintervals are to be approximated using the Weddle's rule, then the Boole's rule is to be used to approximated as much intervals as possible, then the priority is for Simpson's $\mathbf{3/8}$ rule and lastly for any single leftover subinterval we can use the trapezoidal rule.

Starting with the number of sub-intervals $\boldsymbol{n} = \mathbf{9}$ and increasing the count of sub-intervals, the method works fine for $\boldsymbol{n} = \mathbf{9}$ and $\mathbf{10}$, but as we reach $\boldsymbol{n} = \mathbf{11}$, the error increases spontaneously. The method regains its momentum at $\boldsymbol{n} = \mathbf{12, 13}$ and $\mathbf{14}$, but again a spontaneous increase in error occurs at $\boldsymbol{n} = \mathbf{15}$. See Table 1.

This describes that the error starts to raise only where the trapezoidal rule comes into picture. This could be resolved if consider any better option instead of trapezoidal rule and redefine the distribution of sub-intervals for the hybrid.

2.4 A Stable Version of the Modified Algorithm for Combined Numerical Integration

The "Modified Algorithm for Error Reduction in Combined Numerical Integration" (SMA) exhibits accuracy fluctuations when the domain number of sub-intervals, $\boldsymbol{n}$, is increased ($\boldsymbol{n} \geq \mathbf{9}$). Starting with the number of sub-intervals $\boldsymbol{n} = \mathbf{9}$ and increasing the count of sub-intervals, the method works fine for $\boldsymbol{n} = \mathbf{9}$ and $\mathbf{10}$, but the error increases spontaneously at $\boldsymbol{n} = \mathbf{11}$. The algorithm regains momentum for $\boldsymbol{n} = \mathbf{12, 13}$ and $\mathbf{14}$, but a spontaneous increase in error occurs at $\boldsymbol{n} = \mathbf{15}$. This fashion describes that the error starts to raise only where the trapezoidal rule comes into combination (due to its poor order of accuracy). This could be resolved if consider a better option instead of the trapezoidal rule and redefine the distribution of sub-intervals for the algorithm. To overcome the fore-highlighted issue the choice of rules for approximating rules is revised as follows.

It is meaningful to give priority to rules that comes with the highest order of accuracy, Thus, the priority is to be given to the Weddle's rule first, then comes the Six-point rule and lastly, the Simpson's rule. Interestingly, this makes the algorithm more robust, as it can now handle $\mathbf{n} \geq \mathbf{4}$ of sub-intervals.

2.4.1 The Revised Algorithm

i. Choose the number of sub-intervals $\boldsymbol{n} \geq \mathbf{4}$ to divide the interval of integration.

Table 1. Error fashion in the algorithm proposed by Bhatti *et al.* [10] over different number of subintervals

Number of subintervals	Description of Hybrid	Error
9	6W + 3S	Descends
10	6W + 4B	Descends
11	6W + 4B + 1T	Spontaneous rise
12	6W + 4B + 2T	Rises
13	6W + 4B + 3S	Descends
14	6W + 8B	Descends
15	6W + 8B + 1T	Spontaneous rise
16	6W + 8B + 2T	Increase
17	6W + 8B + 3S	Descends
18	6W + 12B	Descends
19	6W + 12B + 1T	Spontaneous rise
20	6W + 12B + 2T	Rises
21	6W + 12B + 3S	Descends
⋮	⋮	⋮

T refers to trapezoidal rule, S to Simpsons 1/3, B to Boole's and W to Weddle's rule

ii. Let $\mathbf{rem}(\boldsymbol{n}, \mathbf{6})$ be the reminder when $\boldsymbol{n}$ is divided by $\mathbf{6}$. The possibilities could be: $\mathbf{rem}(\boldsymbol{n}, \mathbf{6}) = \mathbf{5}, \mathbf{4}, \mathbf{3}, \mathbf{2}, \mathbf{1}, \mathbf{0}$.

- If $\mathbf{rem}(\boldsymbol{n}, \mathbf{6}) = \mathbf{0}$, no sub-intervals are left to be integrated.
- If $\mathbf{rem}(\boldsymbol{n}, \mathbf{6}) = \mathbf{5}$, we use the Six-Point rule to approximate the 5-leftover sub-intervals.
- If $\mathbf{rem}(\boldsymbol{n}, \mathbf{6}) = \mathbf{2}$, we use the Simpson's $\frac{\mathbf{1}}{\mathbf{3}}$ rule to approximate the 2-leftover sub-intervals.
- If $\mathbf{rem}(\boldsymbol{n}, \mathbf{6}) = \mathbf{4}$, we use the composite Simpson's $\frac{\mathbf{1}}{\mathbf{3}}$ rule to approximate the 4-leftover sun-intervals.
- If $\mathbf{rem}(\boldsymbol{n}, \mathbf{6}) = \mathbf{3}$, we reserve 6 sub-interval and approximate the rest $\boldsymbol{n} - \mathbf{6}$ sub-intervals with Weddle's rule. This too gives $\boldsymbol{rem}(\boldsymbol{n} - \mathbf{6}, \mathbf{6}) = \mathbf{3}$, but now with the 6 reserved sub-intervals, we have a total of 9 sub-intervals. These 9 sub-intervals can be sorted as $\mathbf{5} + \mathbf{4}$ and used with the Six-Point and composite Simpson's $\frac{\mathbf{1}}{\mathbf{3}}$ respectively.
- If $\mathbf{rem}(\boldsymbol{n}, \mathbf{6}) = \mathbf{1}$, we again reserve 6 sub-interval. This gives $\mathbf{rem}(\boldsymbol{n} - \mathbf{6}, \mathbf{6}) = \mathbf{1}$, and with the 6 reserved sub-intervals, we have a total of 7 sub-intervals. These 7 sub-intervals can be sorted as $\mathbf{5} + \mathbf{2}$ and used with the Six-Point and Simpson's $\frac{\mathbf{1}}{\mathbf{3}}$ respectively.

iii. Finally, Sum up the segmented integral approximation to get approximate value of given definite integral.

3. RESULTS AND DISCUSSION

Approximate solutions for the three aforementioned examples have been obtained by employing the proposed stable version of the modified algorithm for combined numerical integration (SMA). The computed outcomes are then tested against those obtained through existing methods (EM), like Simpson's 1/3rd or Simpson's 3/8th, depending on the number of segments. Furthermore, we compare the results from the SMA with the previously available algorithm (PMA) by Amanat [5] and the refined version of the modified algorithm (MA) as proposed by Bhatti *et al.* [10].

To evaluate the integrals in the examples, numerical tests have been undertaken. These computations are carried out using MATLAB® R2018b, where the codes are written and executed. The outcomes of the these tests are presented in Tables 2, 3, and 4, depicting the results achieved in figure 1, 2 and 3. Through the computation of both absolute and percentage errors, a comparison of the results is established. Across all instances, the proposed SMA approach consistently showcases its robust stability when places against with EM, PMA, and MA methods.

Example 1.
$\int_{\mathbf{0}}^{\mathbf{1}} \sqrt{\mathbf{1} - \boldsymbol{x}^{\mathbf{2}}}\, \mathbf{d}\boldsymbol{x}$, see Table 2 and Figure 1.

Example 2.
$\int_{\mathbf{1}}^{\mathbf{2}} \boldsymbol{x}\sqrt{\mathbf{1} + \boldsymbol{x}}\, \mathbf{d}\boldsymbol{x}$, see Table 3 and Figure 2.

Example 3.
$\int_{\mathbf{0}}^{\mathbf{1}} \boldsymbol{x}\boldsymbol{e}^{\boldsymbol{x}^{2}}\, \mathbf{d}\boldsymbol{x}$, see Table 4 and Figure 3.

Figures 1, 2, and 3 illustrate graphical representations of Tables 2, 3, and 4. On the graphs, the $\boldsymbol{y}$-axis displays the percentile error when the given integral is computed by EM, PMA, MA, and SMA; while the $\boldsymbol{x}$-axis displays the number of subintervals. Compared to other methods, the proposed approach SMA is observed to have better performance.

4. CONCLUSION

The introduced SMA method is an improved version of the rule proposed by Bhatti et al. [10], incorporating a better the selection of quadrature rule combinations. The research reveals that as the number of subintervals increases, the SMA exhibits significantly greater stability in comparison to the existing composite rules by Amanat [5] and

Table 2. Numerical results from example 1 - EM, PMA and MA in comparison to SMA

	n = 9	n = 10	n = 11
EM	0.774546345	0.776129582	0.777362076
Abs. Error	1.0851e-2	9.2685e-3	8.0360e-3
Per. Error	1.3816 %	1.1801 %	1.0231 %
PMA	0.776456493	0.781754678	0.778798642
Abs. Error	8.9416e-3	3.6434e-3	6.5995e-3
Per. Error	1.1384 %	0.4639 %	0.8402 %
MA	0.780204267	0.782199413	0.778824026
Abs. Error	5.1938e-3	3.1987e-3	6.5741e-3
Per. Error	0.6613 %	0.4072 %	0.8370 %
SMA	0.781128346	0.781754818	0.782341531
Abs. Error	4.2698e-3	3.6433e-3	3.0566e-3
Per. Error	0.5436 %	0.4638 %	0.3891 %

Table 3. Numerical results from example 2 - EM, PMA and MA in comparison to SMA

	n = 9	n = 10	n = 11
EM	2.394714891	2.394609023	2.394530692
Abs. Error	5.5721e-4	4.5134e-4	3.7301e-4
Per. Error	2.3273e-2 %	1.8852e-2 %	1.5580e-2 %
PMA	2.394213311	2.39415769	2.394188088
Abs. Error	5.5635e-5	1.4999e-8	3.0412e-5
Per. Error	2.3238e-3 %	6.3e-7 %	1.2703e-3 %
MA	2.394157718	2.394157674	2.39418808
Abs. Error	4.3000e-8	1.0000e-9	3.0404e-5
Per. Error	1.7960e-6 %	4.0000e-8 %	1.2699e-3 %
SMA	2.394157703	2.394157690	2.394157675
Abs. Error	2.7960e-08	1.5331e-08	1.2720e-10
Per. Error	1.1670e-6 %	6.4000e-8 %	5.3000e-9 %

Table 4. Numerical results from example 3 - EM, PMA and MA in comparison to SMA

	n = 9	n = 10	n = 11
EM	0.862664226	0.862179431	0.861788193
Abs. Error	3.5233e-3	3.0385e-3	2.6473e-3
Per. Error	4.1009e-1 %	3.5367e-1 %	3.0813e-1 %
PMA	0.860034834	0.859147486	0.859737179
Abs. Error	8.9392e-4	6.5720e-6	5.9627e-4
Per. Error	1.0405e-1 %	7.6495e-4 %	6.9402e-2 %
MA	0.859167420	0.859141382	0.859733843
Abs. Error	2.6505e-5	4.680e-7	5.9293e-4
Per. Error	3.0851e-3 %	5.4470e-5 %	6.9014e-2 %
SMA	0.859150653	0.859147358	0.859148341
Abs. Error	9.739e-6	6.4440e-6	7.4270e-6
Per. Error	1.1335e-3 %	7.5005e-4 %	8.6446e-4 %

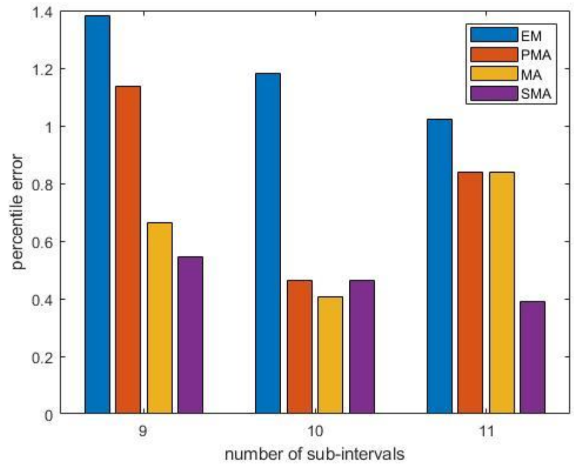

Fig. 1. Results from example 1

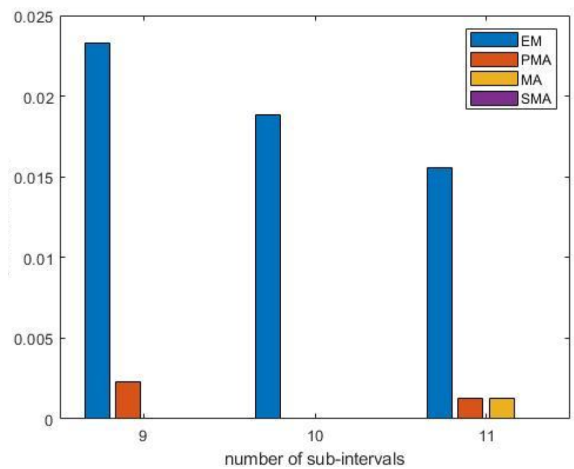

Fig. 2. Results from example 2

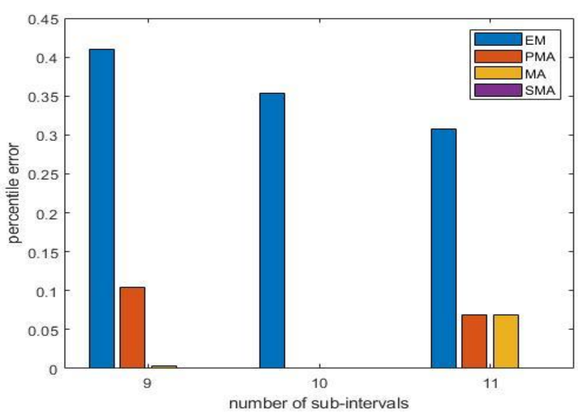

Fig. 3. Results from example 3

Bhatti et al. [10]. The findings demonstrate that the accuracy fluctuations encountered in previous methods are effectively mitigated through the redefined integration rule choices and their combined integration pattern implemented in SMA. This approach emerges as a preferable alternative to the previously employed rules, addressing accuracy concerns with enhanced stability.

5. CONFLICT OF INTEREST

The authors declare no conflict of interest.

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