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Research Article

Some Studies of Multi-Polar Fuzzy Ideals in LA-Semigroups

Fareeha Pervaiz*, Muhammad Shabir, and Muhammad Aslam

Department of Mathematics, Quaid-i-Azam University Islamabad, Pakistan

Abstract: This article's main goal is to investigate the concept of multi-polar fuzzy sets (MPF-sets) in LA-semigroups, which is an extension of bi-polar fuzzy sets (BPF-sets) in LA-semigroups. The main objective of this research is to extend certain significant BPF-set results to MPF-sets results. This article introduces the concepts of multi-polar fuzzy sub LA-semigroups, multi-polar fuzzy quasi-ideals, multi-polar fuzzy bi-ideals, multi-polar fuzzy generalized bi-ideals, and multi-polar fuzzy interior ideals in LA-semigroups. This article also discusses a number of fundamental aspects of multi-polar fuzzy ideals, and we use these aspects to define regular LA-semigroups.

Keywords: Multi-Polar Fuzzy Sub LA-semigroups, Multi-Polar Fuzzy Generalized Bi-Ideals, Multi-Polar Fuzzy Bi-Ideals, Multi-Polar Fuzzy Quasi-Ideals, Multi-Polar Fuzzy Interior Ideals

1. INTRODUCTION

Over the course of the field's evolution, various types of fuzzy set expansions have been developed. The theory of fuzzy sets is well known and has a large variety of applications in many different fields, including decision-making issues, neural networks, artificial intelligence, social sciences, and many more. The use of innovative ideas related to m-polar spherical fuzzy sets for medical diagnosis is investigated by Riaz *et al.* [16]. In the field of multi-criteria decision-making, researchers have recently introduced hybrid structures of MPF-sets to better model uncertainties. The idea of F-set was first represented by Zadeh [13-14]. The structure of fuzzy group is defined by Rosenfeld [12]. Mordeson *et al.* [8] and Kuroki [4] have examined fuzzy semigroups. The application of BPF-sets in decision making is examined by Malik *et al.* [7]. The membership function only ranged over the closed interval [0,1], it is hard to demonstrate the distinctness of irrelevant elements with the contradictory elements in a F-set. On the basis of these observations, the notion of BPF-set was introduced by Lee [5]. The BPF-set is actually an expansion of a F-set whose membership degree lies within the range [-1,1]. In a BPF-set, the associate degree 0 denotes that an element is unrelated to the correlative property, the associate degree from [0,1] denotes that the element partially fulfills the property to a bit extent, and the associate degree from [-1,0] denotes that the element completely fulfills the contrary property to a bit extent [5-6].

A 2-polar -sets and BPF-sets are two algebraic structures. Actually, BPF-set and 2-polar F-set have a natural one-to-one relationship. The BPF-sets can be expanded to MPF-sets by utilizing the concept of a one-to-one relationship. Sometimes, different things have occasionally been observed in various ways. This prompted research into MPF-set. The idea behind this interpretation is predicated on the fact that the given collection contains multi-polar information. MPF-sets have been successful in assigning membership degrees to multiple objects in the context of multi-polar information. In this case, it is important to note that MPF-sets only provide positive degrees of membership for each element, and no negative membership degrees are assumed [1]. Numerous real-world issues involving multiple factors, multiple indices, multiple items, and multiple polarities can be solved using multi-polar F-sets. Multi-polar F-sets can be used for diagnostic data, cooperative games, and decision-making.

A MPF-set can be written as m distinct F-sets, just like the BPF-sets can. As a consequence, every input is expressed by an m-dimensional vector whose entries belongs to [0,1], each represents a degree of confidence. Assume that the collection of context is N = {1,2,3,...,m}. Then, MPF-set will indicate the fulfillment degree of an element with regard to n^{th} context for each $n \in N$ [2]. For example, the F-set "brilliant" can have different interpretations among students in a particular class.

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*Corresponding author: Fareeha Pervaiz <fareeha5531@yahoo.com>

We will give an example to demonstrate it.

Let $Z = \{z_1, z_2, z_3, z_4, z_5\}$ be the collection of 5 students. We shall grade them by a 4-polar F-set based on the following four qualities given below in Table 1.

Table 1. 4 polar fuzzy set

	IQ	Sports	Punctual	Discipline
z_1	1	0	0.8	0.9
z_2	1	0.8	0.5	0.5
z_3	0.5	1	1	0.8
z_4	0.8	0.5	1	0.8
z_5	1	0.5	0.9	0.8

Consequently, we get a 4-polar F-subset $\hat{g} : Z \to [0,1]^4$ of Z such that

$$\hat{g}(z_1) = (1, 0, 0.8, 0.9)$$

$$\hat{g}(z_2) = (1, 0.8, 0.5, 0.5)$$

$$\hat{g}(z_3) = (0.5, 1, 1, 0.8)$$

$$\hat{g}(z_4) = (0.8, 0.5, 1, 0.8)$$

$$\hat{g}(z_5) = (1, 0.5, 0.9, 0.8).$$

Here 1 stands for positive comments, 0.5 for average, and 0 for negative remarks.

In current paper, we define multi-polar fuzzy sub LA-semigroup (MPF-sub LA-semigroup) and multi-polar fuzzy ideals (MPF-ideals) of an LA-semigroup. Besides this, the characterization of regular LA-semigroups by MPF-ideals are presented.

2. PRELIMINARIES

We now illustrate some basic definitions and initial results centred on LA-semigroups that are significant in and of themselves. For the parts that follow, these are necessary. In the present paper, $\hat{S}$ will be denoting an LA-semigroup, unless stated otherwise. The concept of LA-semigroups, was first studied by Kazim and Naseerudin in 1972 [3]. Later on, Yusuf and Mushtaq worked on locally associative LA-semigroups in 1979 [10].

Definition 2.1 If an algebraic structure $(\hat{S},\bullet)$ holds the equation $(r\bullet s)\bullet t = (t\bullet s)\bullet r$ for each r, s, t $\in \hat{S}$, then it is a left almost semigroup (or LA-semigroup) [3].

Some basic definitions which are widely used in LA-semigroup as described below.

If for each $a \in \hat{S}$, $ea = a$, then e in $\hat{S}$ is a left identity. The left identity $e \in \hat{S}$ is unique [9]. Furthermore, if $e \in \hat{S}$, then $\hat{S} = \hat{S}e = e\hat{S}$ and $\hat{S}^2 = \hat{S}$. A left ideal (L-ideal) over $\hat{S}$ is a subset $\hat{I}$ that satisfies $\hat{S}\hat{I} \subseteq \hat{I}$ and right ideal (R-ideal) over $\hat{S}$ if $\hat{I}\hat{S} \subseteq \hat{I}$. $\hat{I}$ is simply termed an ideal (or two-sided) over $\hat{S}$ if $\hat{I}$ is a L-ideal and R-ideal over $\hat{S}$ [11]. A subset $\hat{I}$ over $\hat{S}$ which is non-empty is a sub LA-semigroup over $\hat{S}$ if $\hat{I}^2 \subseteq \hat{I}$. A subset $\hat{I}$ over $\hat{S}$ which is non-empty is a generalized bi-ideal (GB-ideal) over $\hat{S}$ if $(\hat{I}\hat{S})\hat{I} \subseteq \hat{I}$. A sub LA-semigroup $\hat{I}$ over $\hat{S}$ is a bi-ideal (B-ideal) over $\hat{S}$ if $(\hat{I}\hat{S})\hat{I} \subseteq \hat{I}$. A subset $\hat{I}$ over $\hat{S}$ which is non-empty is a quasi-ideal (Q-ideal) over $\hat{S}$ if $\hat{I}\hat{S} \cap \hat{S}\hat{I} \subseteq \hat{I}$. A sub LA-semigroup $\hat{I}$ over $\hat{S}$ is an interior ideal (I-ideal) over $\hat{S}$ if $(\hat{S}\hat{I})\hat{S} \subseteq \hat{I}$.

Definition 2.2 A function $\hat{g} : \hat{S} \to [0,1]$ from $\hat{S}$ into the interval [0,1] is a fuzzy subset (F-subset) of a universe $\hat{S}$.

Some important definitions in F-sets are defined below.

Let $\hat{g}$ be a F-subset over $\hat{S}$. Then the set $\hat{g}_t = \{s \in \hat{S} \mid \hat{g}(s) \geq t\}$ for all $t \in (0,1]$, is named as a level subset over $\hat{S}$.

Let $\hat{g}$ and $\hat{h}$ be any two F-subsets over $\hat{S}$, then $\hat{g} \leq \hat{h}$ means that $\hat{g}(s) \leq \hat{h}(s)$ for each $s \in \hat{S}$. The F-subsets $\hat{g} \wedge \hat{h}$ and $\hat{g} \vee \hat{h}$ of $\hat{S}$ is described as

$(\hat{g} \wedge \hat{h})(s) = \hat{g}(s) \wedge \hat{h}(s)$ and

$(\hat{g} \vee \hat{h})(s) = \hat{g}(s) \vee \hat{h}(s)$ for all $s \in \hat{S}$.

The product $\hat{g} \circ \hat{h}$ is defined as

$$(\hat{g}\circ\hat{h})(s) = \begin{cases} \bigvee_{s=pq}\{\hat{g}(p) \wedge \hat{h}(q)\}, \text{if } \exists\ p, q \in \hat{S} \text{ such that } s = pq \\ 0 \qquad\qquad\qquad\qquad\qquad otherwise \end{cases}$$

for all $s \in \hat{S}$.

A F-subset $\hat{g}$ over $\hat{S}$ is a fuzzy sub LA-semigroup (F-Sub LA-semigroup) over $\hat{S}$ if for every $p, q \in \hat{S}$, $\hat{g}(pq) \geq \hat{g}(p) \wedge \hat{g}(q)$ [15].

For every p, q ∈ $\hat{S}$, a F-subset $\hat{g}$ over $\hat{S}$ is classified as a fuzzy left ideal (FL-ideal) over $\hat{S}$ if $\hat{g}(pq) \geq \hat{g}(q)$ [15].

For every p, q ∈ $\hat{S}$, a F-subset $\hat{g}$ over $\hat{S}$ is classified as a fuzzy right ideal (FR-ideal) over $\hat{S}$ if $\hat{g}(pq) \geq \hat{g}(p)$ [15].

If F-subset $\hat{g}$ is both a FL-ideal and a FR-ideal over $\hat{S}$, so it is a fuzzy ideal (F-ideal) over $\hat{S}$.

A F-subset $\hat{g}$ over $\hat{S}$ is a fuzzy quasi-ideal (FQ-ideal) over $\hat{S}$ if $(\hat{g} \circ \delta) \wedge (\delta \circ \hat{g}) \leq \hat{g}$. Here, δ is the F-subset over $\hat{S}$ which maps each element of $\hat{S}$ on 1, that is δ is the characteristic function over $\hat{S}$ [15].

A F-subset $\hat{g}$ over $\hat{S}$ is a fuzzy generalized bi-ideal (FGB-ideal) over $\hat{S}$ if $\hat{g}((pq)r) \geq \hat{g}(p) \wedge \hat{g}(r)$ for each p, q, r ∈ $\hat{S}$ [15].

A F-Sub LA-semigroup $\hat{g}$ over $\hat{S}$ is known as a fuzzy bi-ideal (FB-ideal) over $\hat{S}$ if $\hat{g}((pq)r) \geq \hat{g}(p) \wedge \hat{g}(r)$ for each p, q , r ∈ $\hat{S}$ [15].

A F-Sub LA-semigroup $\hat{g}$ over $\hat{S}$ is a fuzzy interior-ideal (FI-ideal) over $\hat{S}$ if for all p, q, r ∈ $\hat{S}$, $\hat{g}((pq)r) \geq \hat{g}(q)$ [15].

3. RESULTS AND DISCUSSION

Now, we define some notions and present our main results regarding multi-polar fuzzy ideals in $\hat{S}$.

Definition 3.1 [1] Multi-polar fuzzy subset over $\hat{S}$ is a mapping $\hat{g} : \hat{S} \rightarrow [0,1]^m$.

MPF-set is represented by the m-tuple $\hat{g} = (\hat{g}_1,\hat{g}_2,...,\hat{g}_m)$, consists of mappings $\hat{g}_n : \hat{S} \rightarrow [0,1]$ for each n ∈ {1,2,3,...,m}. The collection of all MPF-subsets of $\hat{S}$, is represented as m($\hat{S}$). We define a relation ≤ on m($\hat{S}$) in the following manner:

For any two MPF-subsets $\hat{g} = (\hat{g}_1,\hat{g}_2,...,\hat{g}_m)$ and $\hat{h} = (\hat{h}_1,\hat{h}_2,...,\hat{h}_m)$ of an LA-semigroup $\hat{S}$, $\hat{g} \leq \hat{h}$ means that $\hat{g}_n(s) \leq \hat{h}_n(s)$ for each s ∈ $\hat{S}$ and n ∈ {1,2,3,...,m}.

The symbols $\hat{g} \wedge \hat{h}$ and $\hat{g} \vee \hat{h}$ denotes the following MPF-subsets over $\hat{S}$.

$(\hat{g} \wedge \hat{h})(s) = \hat{g}(s) \wedge \hat{h}(s)$ and $(\hat{g} \vee \hat{h})(s) = \hat{g}(s) \vee \hat{h}(s)$ that is $(\hat{g}_n \wedge \hat{h}_n)(s) = \hat{g}_n(s) \wedge \hat{h}_n(s)$ and $(\hat{g}_n \vee \hat{h}_n)(s) = \hat{g}_n(s) \vee \hat{h}_n(s)$ for each s ∈ $\hat{S}$ and n ∈ {1,2,3,...,m}.

Let $\hat{g} = (\hat{g}_1,\hat{g}_2,...,\hat{g}_m)$ and $\hat{h} = (\hat{h}_1,\hat{h}_2,...,\hat{h}_m)$ be any two MPF-subsets over $\hat{S}$.

The product $\hat{g} \circ \hat{h} = (\hat{g}_1 \circ \hat{h}_1, \hat{g}_2 \circ \hat{h}_2,..,\hat{g}_n \circ \hat{h}_n)$ is defined as

$(\hat{g}_n \circ \hat{h}_n) =$

$$\begin{cases} \bigvee_{s=pq}\{\hat{g}_n(p) \wedge \hat{h}_n(q)\}, \text{if } s = pq \text{ for some } p, q \in \hat{S} \\ 0 \qquad\qquad\qquad\qquad otherwise \end{cases}$$

for every n ∈ {1,2,3,...,m}.

For m = 3, the following example illustrates the product of MPF-subsets $\hat{g}$ and $\hat{h}$ over $\hat{S}$.

Example 3.1 Let the LA-semigroup $\hat{S}$ = {u, v, w} with the binary operation "·" is defined as (Table 2):

Table 2. LA-semigroup

•	U	v	W
U	U	u	U
V	U	u	U
w	V	v	U

We define 3-polar fuzzy subsets $\hat{g} = (\hat{g}_1,\hat{g}_2,\hat{g}_3)$ and $\hat{h} = (\hat{h}_1,\hat{h}_2,\hat{h}_3)$ of $\hat{S}$ as follows:

$\hat{g}(u) = (0.1,0.2,0.1)$, $\hat{g}(v) = (0,0,0)$, $\hat{g}(w) = (0.2,0.3,0.4)$

and

$\hat{h}(u) = (0,0,0)$, $\hat{h}(v) = (0,0.1,0.2)$, $\hat{h}(w) = (0.3,0,0.4)$.

By definition,

$(\hat{g}_1 \circ \hat{h}_1)(u) = 0.2$, $(\hat{g}_1 \circ \hat{h}_1)(v) = 0$, $(\hat{g}_1 \circ \hat{h}_1)(w) = 0$

$(\hat{g}_2 \circ \hat{h}_2)(u) = 0.1$, $(\hat{g}_2 \circ \hat{h}_2)(v) = 0.1$, $(\hat{g}_2 \circ \hat{h}_2)(w) = 0$

$(\hat{g}_3 \circ \hat{h}_3)(u) = 0.4$, $(\hat{g}_3 \circ \hat{h}_3)(v) = 0.2$, $(\hat{g}_3 \circ \hat{h}_3)(w) = 0$

So, the product of $\hat{g} = (\hat{g}_1,\hat{g}_2,\hat{g}_3)$ and $\hat{h} = (\hat{h}_1,\hat{h}_2,\hat{h}_3)$ is defined by

$(\hat{g} \circ \hat{h})(u) = (0.2,0.1,0.4)$,

$(\hat{g} \circ \hat{h})(v) = (0,0.1,0.2)$

$(\hat{g} \circ \hat{h})(w) = (0,0,0)$.

Definition 3.2 Consider $\hat{g} = (\hat{g}_1,\hat{g}_2,...,\hat{g}_m)$ be a MPF-subset over $\hat{S}$.

(1) Let $\hat{g}_t = \{x \in \hat{S} \mid \hat{g}(x) \geq t\}$ be defined for each t and $t = (t_1,t_2,...,t_m) \in (0,1]^m$, such that $\hat{g}_n(x) \geq t_n$ for each $n \in \{1,2,3,...,m\}$. We name $\hat{g}_t$ a t-cut or sometimes a level set. This means $\hat{g}_t = \cap_{k=1}^{m}(\hat{g}_n)t_n$.

Definition 3.3 A multi-polar fuzzy subset $\hat{g} = (\hat{g}_1,\hat{g}_2,...,\hat{g}_m)$ over $\hat{S}$ is a multi-polar fuzzy sub LA-semigroup (MPF-sub LA-semigroup) over $\hat{S}$ if $\hat{g}(xy) \geq \min\{\hat{g}(x), \hat{g}(y)\}$ for every $x,y \in \hat{S}$, that is $\hat{g}_n(xy) \geq \min\{\hat{g}_n(x), \hat{g}_n(y)\}$ for each $n \in \{1,2,3,...,m\}$.

Definition 3.4 A multi-polar fuzzy subset $\hat{g} = (\hat{g}_1,\hat{g}_2,...,\hat{g}_m)$ over $\hat{S}$ is a multi-polar fuzzy left ideal (MPFL-ideal) over $\hat{S}$ if for each $x, y \in \hat{S}$, $\hat{g}(xy) \geq \hat{g}(y)$, that is $\hat{g}_n(xy) \geq \hat{g}_n(y)$ and multi-polar fuzzy right ideal (MPFR-ideal) over $\hat{S}$ if for each $x, y \in \hat{S}$, $\hat{g}(xy) \geq \hat{g}(x)$, that is $\hat{g}_n(xy) \geq \hat{g}_n(x)$ for each $n \in \{1,2,3,...,m\}$.

A MPF-subset $\hat{g}$ over $\hat{S}$ is considered a MPF-ideal over $\hat{S}$ if it satisfies the conditions of being a multi-polar fuzzy left ideal (MPFL-ideal) and a multi-polar fuzzy right ideal (MPFR-ideal) over $\hat{S}$.

The next example is of 3-polar fuzzy two-sided ideal over $\hat{S}$.

Example 3.2 Consider $\hat{S} = \{r,s,t,u,v\}$ be an LA-semigroup under the binary operation"·" defined below in Table 3.

Table 3. LA-semigroup

•	R	s	t	u	v
R	R	r	r	r	r
S	R	s	s	s	s
T	R	s	u	v	t
U	R	s	t	u	v
V	R	s	v	t	u

We define a 3-polar fuzzy subset $\hat{g} = (\hat{g}_1,\hat{g}_2,\hat{g}_3)$ of $\hat{S}$ as follows:

$\hat{g}(r) = (0.8,0.8,0.7)$, $\hat{g}(s) = (0.7,0.6, 0.5)$,

$\hat{g}(t) = (0.6,0.4, 0.2)$, $\hat{g}(u) = (0.6, 0.4,0.2)$ and

$\hat{g}(v) = (0.6, 0.4, 0.2)$.

Clearly, $\hat{g} = (\hat{g}_1,\hat{g}_2,\hat{g}_3)$ is both a 3-polar FL-ideal and a 3-polar FR-ideal over $\hat{S}$. Hence $\hat{g}$ is a 3-polar fuzzy two-sided ideal over $\hat{S}$.

Definition 3.5 Let $\varphi \neq \hat{A} \subseteq \hat{S}$, where $\hat{S}$ be an LA-semigroup. Subsequently, the multi-polar characteristic function

$\hat{C}_{\hat{A}} : X \rightarrow [0,1]^m$ of $\hat{A}$ is described as

$$\hat{C}_{\hat{A}}(x) = \begin{cases} (1,1,\ldots,1) \text{ m} - \text{tuple for } x \in \hat{A} \\ (0,0,\ldots,0) \text{ m} - \text{tuple for } x \notin \hat{A} \end{cases}$$

Lemma 3.1 For any two subsets $\hat{A} \neq \varphi$ and $\hat{B} \neq \varphi$ of an LA-semigroup $\hat{S}$. The subsequent equalities are hold.

(1) $\hat{C}_{\hat{A}} \wedge \hat{C}_{\hat{B}} = \hat{C}_{\hat{A}\cap\hat{B}}$.

(2) $\hat{C}_{\hat{A}} \vee \hat{C}_{\hat{B}} = \hat{C}_{\hat{A}\cup\hat{B}}$

(3) $\hat{C}_{\hat{A}} \circ \hat{C}_{\hat{B}} = \hat{C}_{\hat{A}\hat{B}}$

Proof. (1) Let $\hat{A} \neq \varphi$ and $\hat{B} \neq \varphi$ be two subsets over $\hat{S}$. We examine the four cases as below,

Case 1: Consider $x \in \hat{A} \cap \hat{B}$. Then, $\hat{C}_{\hat{A}\cap\hat{B}}(x) = (1,1,...,1)$. Also $x \in \hat{A} \cap \hat{B}$ implies $x \in \hat{A}$ and $x \in \hat{B}$. Hence, $\hat{C}_{\hat{A}}(x) = (1,1,...,1)$ and $\hat{C}_{\hat{B}}(x) = (1,1,...,1)$. This implies that $(\hat{C}_{\hat{A}} \wedge \hat{C}_{\hat{B}})(x) = \hat{C}_{\hat{A}}(x) \wedge \hat{C}_{\hat{B}}(x) = (1,1,...,1)$. Thus, $\hat{C}_{\hat{A}} \wedge \hat{C}_{\hat{B}} = \hat{C}_{\hat{A}\cap\hat{B}}$.

Case 2: Consider $x \notin \hat{A} \cap \hat{B}$. Then $\hat{C}_{\hat{A}\cap\hat{B}}(x) = (0,0,...,0)$. As $x \notin \hat{A} \cap \hat{B}$ thus $x \notin \hat{A}$ or $x \notin \hat{B}$. As a result, it follows that $\hat{C}_{\hat{A}}(x) = (0,0,...,0)$ or $\hat{C}_{\hat{B}}(x) = (0,0,...,0)$. Thus, $(\hat{C}_{\hat{A}} \wedge \hat{C}_{\hat{B}})(x) = \hat{C}_{\hat{A}}(x) \wedge \hat{C}_{\hat{B}}(x) = (0,0,...,0)$. Therefore $\hat{C}_{\hat{A}} \wedge \hat{C}_{\hat{B}} = \hat{C}_{\hat{A}\cap\hat{B}}$.

(2) Consider $\hat{A}$ and $\hat{B}$ denote non-empty subsets of $\hat{S}$.

Case 1: Let $x \in \hat{A} \cup \hat{B}$. Then, $\hat{C}_{\hat{A}\cup\hat{B}}(x) = (1,1,...,1)$. Since $x \in \hat{A} \cup \hat{B}$ implies $x \in \hat{A}$ or $x \in \hat{B}$. Hence, $\hat{C}_{\hat{A}}(x) = (1,1,...,1)$ or $\hat{C}_{\hat{B}}(x) = (1,1,...,1)$. As a result, it follows that $(\hat{C}_{\hat{A}} \vee \hat{C}_{\hat{B}})(x) = \hat{C}_{\hat{A}}(x) \vee \hat{C}_{\hat{B}}(x) = (1,1,...,1)$. Thus, $\hat{C}_{\hat{A}} \vee \hat{C}_{\hat{B}} = \hat{C}_{\hat{A}\cup\hat{B}}$.

Case 2: Let $x \notin \hat{A} \cup \hat{B}$. Then $\hat{C}_{\hat{A}\cup\hat{B}}(x) = (0,0,...,0)$. Since $x \notin \hat{A} \cup \hat{B}$, we get $x \notin \hat{A}$ and $x \notin B$. This implies that $\hat{C}_{\hat{A}}(x) = (0,0,...,0)$ and $\hat{C}_{\hat{B}}(x) = (0,0,...,0)$. Thus, $(\hat{C}_{\hat{A}} \vee \hat{C}_{\hat{B}})(x) = \hat{C}_{\hat{A}}(x) \vee \hat{C}_{\hat{B}}(x) = (0,0,...,0)$. Hence $\hat{C}_{\hat{A}} \vee \hat{C}_{\hat{B}} = \hat{C}_{\hat{A}\cup\hat{B}}$.

(3) Let $\hat{A} \neq \varphi$ and $\hat{B} \neq \varphi$ be subsets over $\hat{S}$.

Case 1: Let x ∈ $\hat{A}\hat{B}$, which implies that x = ab for a ∈ $\hat{A}$ and b ∈ $\hat{B}$. Thus $\hat{C}_{\hat{A}\hat{B}}$(x) = (1,1,...,1). Since a ∈ $\hat{A}$ and b ∈ $\hat{B}$, we have $\hat{C}_{\hat{A}}$(a) = (1,1,...,1) and $\hat{C}_{\hat{B}}$(b) = (1,1,...,1). Now,

$$(\hat{C}_{\hat{A}}\circ\hat{C}_{\hat{B}})(x) = \vee_{x=uv}\{\hat{C}_{\hat{A}}(u)\wedge\hat{C}_{\hat{B}}(v)\}$$

$$\geq \hat{C}_{\hat{A}}(a)\wedge\hat{C}_{\hat{B}}(b)$$

$$= (1,1,...,1)$$

Thus, $\hat{C}_{\hat{A}} \circ \hat{C}_{\hat{B}} = \hat{C}_{\hat{A}\hat{B}}$.

Case 2: Let x ∉ $\hat{A}\hat{B}$. This implies that $\hat{C}_{\hat{A}\hat{B}}$(x) = (0,0,...,0). Because x ≠ ab for each a ∈ $\hat{A}$ and b ∈ $\hat{B}$. So, $(\hat{C}_{\hat{A}}\circ\hat{C}_{\hat{B}})(x) = \vee_{x=ab}\{\hat{C}_{\hat{A}}(a)\wedge\hat{C}_{\hat{B}}(b)\}$ = (0,0,...,0).

Hence $\hat{C}_{\hat{A}} \circ \hat{C}_{\hat{B}} = \hat{C}_{\hat{A}\hat{B}}$.

Lemma 3.2 Consider $\hat{L} \neq \varphi$ be a subset of $\hat{S}$. So the subsequent assertions hold.

(1) $\hat{L}$ is a sub LA-semigroup over $\hat{S}$ iff $\hat{C}_{\hat{L}}$ is a multi-polar fuzzy sub LA-semigroup over $\hat{S}$.

(2) $\hat{L}$ is a left (right, two-sided) ideal over $\hat{S}$ iff $\hat{C}_{\hat{L}}$ is a multi-polar fuzzy left (right, two-sided) ideal over $\hat{S}$.

Proof. (1) Consider $\hat{L}$ is a sub LA-semigroup over $\hat{S}$. We claim that

$\hat{C}_{\hat{L}}(xy) \geq \hat{C}_{\hat{L}}(x) \wedge \hat{C}_{\hat{L}}(y)$ for every x, y ∈ $\hat{S}$. We examine the four cases as below,

Case 1 : Let x, y ∈ $\hat{L}$. So, $\hat{C}_{\hat{L}}$(x) = $\hat{C}_{\hat{L}}$(y) = (1,1,...,1). Since $\hat{L}$ is a sub LA-semigroup over $\hat{S}$, so xy ∈ $\hat{L}$ it follows that $\hat{C}_{\hat{L}}$(xy) = (1,1,...,1). Hence $\hat{C}_{\hat{L}}(xy) \geq \hat{C}_{\hat{L}}(x) \wedge \hat{C}_{\hat{L}}(y)$.

Case 2 : Consider x ∈ $\hat{L}$, y ∉ $\hat{L}$. Then, $\hat{C}_{\hat{L}}$(x) = (1,1,...,1) and $\hat{C}_{\hat{L}}$(y) = (0,0,...,0). So, $\hat{C}_{\hat{L}}$x) ∧ $\hat{C}_{\hat{L}}$(y) = (0,0,...,0). But $\hat{C}_{\hat{L}}$(xy) ≥ (0,0,...,0). Thus $\hat{C}_{\hat{L}}$(xy) ≥ $\hat{C}_{\hat{L}}$(x) ∧$\hat{C}_{\hat{L}}$(y).

Case 3 : Consider x, y ∉ $\hat{L}$. Then, $\hat{C}_{\hat{L}}$(x) = $\hat{C}_{\hat{L}}$(y) = (0,0,...,0). Clearly, $\hat{C}_{\hat{L}}$(xy) ≥ (0,0,...,0) = $\hat{C}_{\hat{L}}$(x) ∧ $\hat{C}_{\hat{L}}$(y).

Case 4 : Consider x ∉ $\hat{L}$, y ∈ $\hat{L}$. Then, $\hat{C}_{\hat{L}}$(x) = (0,0,...,0) and $\hat{C}_{\hat{L}}$(y) = (1,1,...,1). Clearly, $\hat{C}_{\hat{L}}$(xy) ≥ (0,0,...,0) = $\hat{C}_{\hat{L}}$(x) ∧ $\hat{C}_{\hat{L}}$(y).

Conversely, let $\hat{C}_{\hat{L}}$ is a MPF-sub LA-semigroup over $\hat{S}$ and x, y ∈ $\hat{L}$. Then, $\hat{C}_{\hat{L}}$(x) = $\hat{C}_{\hat{L}}$y) = (1,1,...,1). By definition, $\hat{C}_{\hat{L}}$(xy) ≥ $\hat{C}_{\hat{L}}$(x) ∧$\hat{C}_{\hat{L}}$(y) = (1,1,...,1) ∧ (1,1,...,1) = (1,1,...,1), we have $\hat{C}_{\hat{L}}$(xy) = (1,1,...,1). This implies that xy ∈ $\hat{L}$, that is $\hat{L}$ is a sub LA-semigroup over $\hat{S}$.

(2) Suppose that $\hat{L}$ is a L-ideal over $\hat{S}$. We show that $\hat{C}_{\hat{L}}(xy) \geq \hat{C}_{\hat{L}}(y)$ for every x,y ∈ $\hat{S}$. We examine the two cases as below,

Case 1 : Consider y ∈ $\hat{L}$ and x ∈ $\hat{S}$. Then, $\hat{C}_{\hat{L}}$(y) = (1,1,...,1). As $\hat{L}$ is a L-ideal over $\hat{S}$, so xy ∈ $\hat{L}$ implies that $\hat{C}_{\hat{L}}$(xy) = (1,1,...,1). Hence $\hat{C}_{\hat{L}}$(xy) ≥ $\hat{C}_{\hat{L}}$(y).

Case 2 : Let y ∉ $\hat{L}$ and x ∈ $\hat{S}$. Then, $\hat{C}_{\hat{L}}$(y) = (0,0,...,0). Clearly, $\hat{C}_{\hat{L}}$(xy) ≥ $\hat{C}_{\hat{L}}$(y).

Conversely, let $\hat{C}_{\hat{L}}$ is a MPFL-ideal over $\hat{S}$. Consider that x ∈ $\hat{S}$ and y ∈ $\hat{L}$. Thus, $\hat{C}_{\hat{L}}$(y) = (1,1,...,1). By definition, $\hat{C}_{\hat{L}}$(xy) ≥ $\hat{C}_{\hat{L}}$(y) = (1,1,...,1), we get

$\hat{C}_{\hat{L}}$(xy) = (1,1,...,1). So xy ∈ $\hat{L}$, as a result $\hat{L}$ is a L-ideal over $\hat{S}$.

Likewise, we can demonstrate that $\hat{L}$ is a R-ideal over $\hat{S}$ iff $\hat{C}_{\hat{L}}$ is a MPFR-ideal over $\hat{S}$. Thus $\hat{L}$ is a two-sided ideal over $\hat{S}$ iff $\hat{C}_{\hat{L}}$ is a multi-polar fuzzy two-sided ideal over $\hat{S}$.

Lemma 3.3 Consider $\hat{g} = (\hat{g}_1, \hat{g}_2, ..., \hat{g}_m)$ be a MPF-subset over $\hat{S}$. Then the subsequent assertions hold.

(1) $\hat{g}$ is a MPF-sub LA-semigroup over $\hat{S}$ iff

$\hat{g} \circ \hat{g} \leq \hat{g}$.

(2) $\hat{g}$ is a MPFL-ideal over $\hat{S}$ iff

$\delta \circ \hat{g} \leq \hat{g}$.

(3) $\hat{g}$ is a MPFR-ideal over $\hat{S}$ iff

$\hat{g} \circ \delta \leq \hat{g}$.

(4) $\hat{g}$ is a multi-polar fuzzy two sided over $\hat{S}$

iff $\delta \circ \hat{g} \leq \hat{g}$ and $\hat{g} \circ \delta \leq \hat{g}$.

Here, δ represents the MPF-subset over $\hat{S}$ that maps every element of $\hat{S}$ to (1,1,...,1).

Proof. (1) Consider that $\hat{g} = (\hat{g}_1, \hat{g}_2, ..., \hat{g}_m)$ be a MPF-sub LA-semigroup over $\hat{S}$, i.e. $\hat{g}_n(xy) \geq \hat{g}_n(x) \wedge \hat{g}_n(y)$ for all n ∈{1,2,3,...,m}. Let a ∈ $\hat{S}$. If a ≠ bc for any b, c ∈ $\hat{S}$, so that $(\hat{g} \circ \hat{g})(a) = 0$. Hence, $(\hat{g} \circ \hat{g})(a) \leq \hat{g}(a)$. But if a = xy for x,y ∈ $\hat{S}$, then

$(\hat{g}_n \circ \hat{g}_n)(a) = \bigvee_{a=xy}\{\hat{g}_n(x) \wedge \hat{g}_n(y)\}$

$\leq \bigvee_{a=xy}\{\hat{g}_n(xy)\}$

$= \hat{g}_n(a)$ for every $n \in \{1,2,3,...,m\}$.

Therefore $\hat{g} \circ \hat{g} \leq \hat{g}$.

Conversely, assume that $(\hat{g} \circ \hat{g}) \leq \hat{g}$ and $x,y \in \hat{S}$. Then

$\hat{g}_n(xy) \geq (\hat{g}_n \circ \hat{g}_n)(xy)$

$= \bigvee_{xy=uv}\{\hat{g}_n(u) \wedge \hat{g}_n(v)\}$

$\geq \hat{g}_n(x) \wedge \hat{g}_n(y)$ for each $n \in \{1,2,3,...,m\}$.

Hence $\hat{g}(xy) \geq \hat{g}(x) \wedge \hat{g}(y)$. Thus $\hat{g}$ is a MPF-sub LA-semigroup over $\hat{S}$.

(2) Let $\hat{g} = (\hat{g}_1,\hat{g}_2,...,\hat{g}_m)$ be a MPFL-ideal over $\hat{S}$, i.e. $\hat{g}_n(xy) \geq \hat{g}_n(y)$ for every $x,y \in \hat{S}$ and $n \in \{1,2,3,...,m\}$. Consider $a \in \hat{S}$. If $a \neq bc$ for $b,c \in \hat{S}$, therefore

$(\delta\circ\hat{g})(a) = 0$. Hence, $\delta\circ\hat{g} \leq \hat{g}$. But if $a = xy$ for $x, y \in \hat{S}$, then

$(\delta_n \circ \hat{g}_n)(a) = \bigvee_{a=xy}\{\delta_n(x) \wedge \hat{g}_n(y)\}$

$= \bigvee_{a=xy}\{\hat{g}_n(y)\}$

$\leq \bigvee_{a=xy}\hat{g}_n(xy)$

$= \hat{g}_n(a)$ for all $n \in\{1,2,3,...,m\}$.

Thus $\delta \circ \hat{g} \leq \hat{g}$.

Conversely, assume that $(\delta \circ \hat{g}) \leq \hat{g}$ and $x,y \in \hat{S}$. Then

$\hat{g}_n(xy) \geq (\delta_n \circ\hat{g}_n)(xy)$

$= \bigvee_{xy=uv}\{\delta_n (u) \wedge \hat{g}_n(v)\}$

$\geq \{\delta_n (x) \wedge \hat{g}_n(y)\}$

$= \hat{g}_n(y)$ for all $n \in \{1,2,3,...,m\}$.

Hence $\hat{g}(xy) \geq \hat{g}(y)$. Thus $\hat{g}$ is a MPFL-ideal over $\hat{S}$.

(3) It can be proved on the same lines of (2).

(4) This can be proved by using equations (2) and (3).

Lemma 3.4 The subsequent statements hold for an LA-semigroup $\hat{S}$.

(1) Consider that $\hat{g} = (\hat{g}_1,\hat{g}_2,...,\hat{g}_m)$ and $\hat{h} = (\hat{h}_1,\hat{h}_2,...,\hat{h}_m)$ be two MPF-sub LA-semigroups over $\hat{S}$. Thus $\hat{g} \wedge \hat{h}$ is also a MPF-sub LA-semigroup over $\hat{S}$.

(2) Consider that $\hat{g} = (\hat{g}_1,\hat{g}_2,...,\hat{g}_m)$ and $\hat{h} = (\hat{h}_1,\hat{h}_2,...,\hat{h}_m)$ be two multi-polar fuzzy left (right, two-sided) ideals over $\hat{S}$. Then $\hat{g} \wedge \hat{h}$ is also a multi-polar fuzzy left (right, two-sided) ideal over $\hat{S}$.

Proof. Assume that $\hat{g} = (\hat{g}_1,\hat{g}_2,...,\hat{g}_m)$ and $\hat{h} = (\hat{h}_1,\hat{h}_2,...,\hat{h}_m)$ be two MPF-sub LA-semigroups over $\hat{S}$. Then

$(\hat{g}_n \wedge \hat{h}_n)(xy) = \hat{g}_n(xy) \wedge \hat{h}_n(xy)$

$\geq (\hat{g}_n(x) \wedge \hat{g}_n(y)) \wedge (\hat{h}_n(x) \wedge \hat{h}_n(y))$

$= (\hat{g}_n(x) \wedge \hat{h}_n(x)) \wedge (\hat{g}_n(y) \wedge \hat{h}_n(y))$

$= (\hat{g}_n \wedge \hat{h}_n)(x) \wedge (\hat{g}_n \wedge \hat{h}_n)(y)$

for each $n \in \{1,2,3,...,m\}$.

Thus, $\hat{g} \wedge \hat{h}$ is a MPF-sub LA-semigroup over $\hat{S}$.

Similar methods can be applied to prove other cases.

Proposition 3.1 Suppose that $\hat{g} = (\hat{g}_1,\hat{g}_2,...,\hat{g}_m)$ be a MPF-subset over $\hat{S}$. Then $\hat{g}$ is a MPF-sub LA-semigroup (left, right, two-sided ideal) over $\hat{S}$ iff $\hat{g}_t = \{x \in \hat{S} \mid \hat{g}(x) \geq t\} \neq \varphi$ is a sub LA-semigroup (left, right, two-sided ideal) over $\hat{S}$ for every $t = (t_1,t_2,...,t_m) \in (0,1]^m$.

Proof. Suppose $\hat{g}$ is a MPF-sub LA-semigroup over $\hat{S}$.Consider $x, y \in \hat{g}_t$. Then $\hat{g}_n(x) \geq t_n$ and $\hat{g}_n(y) \geq t_n$ for each $n \in \{1,2,3,...,m\}$. As $\hat{g}$ is a MPF-sub LA-semigroup over $\hat{S}$, we have $\hat{g}_n(xy) \geq \hat{g}_n(x) \wedge \hat{g}_n(y) \geq t_n \wedge t_n = t_n$ for every $n \in \{1,2,3,...,m\}$. Thus $xy \in \hat{g}_t$. So $\hat{g}_t$ is a sub LA-semigroup over $\hat{S}$.

Conversely, assume that $\hat{g}_t \neq \varphi$ is a sub LA-semigroup over $\hat{S}$. On contrary, let $\hat{g}$ is not a MPF-sub LA-semigroup over $\hat{S}$. Consider $x, y \in \hat{S}$ with $\hat{g}_n(xy) < \hat{g}_n(x) \wedge \hat{g}_n(y)$ for $n \in \{1,2,3,..,m\}$. Take $t_n = \hat{g}_n(x) \wedge \hat{g}_n(y)$ for every $n \in \{1,2,3,...,m\}$. Then $x, y \in \hat{g}_t$ but $xy \notin \hat{g}_t$, this contradicts the hypothesis. Hence $\hat{g}(xy) \geq \hat{g}(x) \wedge \hat{g}(y)$. Thus $\hat{g}$ is a MPF-sub LA-semigroup over $\hat{S}$.

Similar methods can be applied to prove other cases.

Next, we define the multi-polar fuzzy generalized bi-ideal (MPFGB-ideal) over $\hat{S}$.

Definition 3.6 A MPF-subset $\hat{g} = (\hat{g}_1,\hat{g}_2,...,\hat{g}_m)$ over $\hat{S}$ is considered a MPFGB-ideal over $\hat{S}$ if for each x, y, z ∈ $\hat{S}$, $\hat{g}((xy)z) \geq \hat{g}(x) \wedge \hat{g}(z)$, that is $\hat{g}_n((xy)z) \geq \hat{g}_n(x) \wedge \hat{g}_n(z)$ for each n ∈ {1,2,...,m}.

Lemma 3.5 A subset $\hat{G}$ over $\hat{S}$ which is non-empty is a GB-ideal over $\hat{S}$ iff $\hat{C}_{\hat{G}}$ the multi-polar characteristic function of $\hat{G}$ is a MPFGB-ideal over $\hat{S}$.

Proof. It can be showed on the same lines of Lemma 3.2.

Lemma 3.6 A MPF-subset $\hat{g}$ over $\hat{S}$ is a MPFGB-ideal over $\hat{S}$ iff $(\hat{g} \circ \delta) \circ \hat{g} \leq \hat{g}$.

Proof. Suppose $\hat{g} = (\hat{g}_1,\hat{g}_2,...,\hat{g}_m)$ be a MPFGB-ideal over $\hat{S}$, i.e. $\hat{g}_n((xy)z) \geq \hat{g}_n(x) \wedge \hat{g}_n(z)$ for each n ∈ {1,2,3,...,m} and x, y, z ∈ $\hat{S}$. Consider a ∈ $\hat{S}$. If a ≠ bc for some b, c ∈ $\hat{S}$ thus $((\hat{g} \circ \delta) \circ \hat{g})(a) = 0$. Therefore, $(\hat{g} \circ \delta) \circ \hat{g} \leq \hat{g}$. But if a = xy for some x,y ∈ $\hat{S}$. Thus for every n ∈ {1,2,3,...,m}.

$$((\hat{g}_n \circ \delta_n) \circ \hat{g}_n))(a) = \vee_{a=xy}\{(\hat{g}_n \circ \delta_n)(x) \wedge \hat{g}_n(y)\}$$

$$= \vee_{a=xy}\{\vee_{x=uv}\{\hat{g}_n(u) \wedge \delta_n(v)\} \wedge \hat{g}_n(y)\}$$

$$= \vee_{a=xy}\{\vee_{x=uv}\{\hat{g}_n(u) \wedge \hat{g}_n(y)\}\}$$

$$\leq \vee_{a=xy}\{\vee_{x=uv}\hat{g}_n((uv)y)\}$$

$$= \vee_{a=xy}\{\hat{g}_n(xy)\}$$

$$= \hat{g}_n(a) \text{ for all } n \in \{1,2,3,...,m\}.$$

So $(\hat{g} \circ \delta) \circ \hat{g} \leq \hat{g}$.

Conversely, let $(\hat{g} \circ \delta) \circ \hat{g} \leq \hat{g}$ and x, y, z ∈ $\hat{S}$. Then

$$\hat{g}_n((xy)z) \geq ((\hat{g}_n \circ \delta_n) \circ \hat{g}_n)((xy)z)$$

$$= \vee_{(xy)z=uv}\{(\hat{g}_n \circ \delta_n)(u) \wedge \hat{g}_n(v)\}$$

$$\geq (\hat{g}_n \circ \delta_n)(xy) \wedge \hat{g}_n(z)$$

$$= \vee_{xy=ab}\{\hat{g}_n(a) \wedge \delta_n(b)\} \wedge \hat{g}_n(z)$$

$$\geq \{\hat{g}_n(x) \wedge \delta_n(y)\} \wedge \hat{g}_n(z)$$

$$= \hat{g}_n(x) \wedge \hat{g}_n(z) \text{ for every } n \in \{1,2,3,...,m\}.$$

Hence, $\hat{g}((xy)z) \geq \hat{g}(x) \wedge \hat{g}(z)$. Thus $\hat{g}$ is a MPFGB-ideal over $\hat{S}$.

Proposition 3.2 Consider $\hat{g} = (\hat{g}_1,\hat{g}_2,...,\hat{g}_m)$ is a multi-polar fuzzy subset over $\hat{S}$. Thus $\hat{g}$ is a MPFGB-ideal over $\hat{S}$ iff $\hat{g}_t = \{x \in \hat{S} \mid \hat{g}(x) \geq t\} \neq \varphi$ is a GB-ideal over $\hat{S}$ for every $t = (t_1,t_2,t_3,...,t_m) \in (0,1]^m$.

Proof. Suppose that $\hat{g}$ be a MPFGB-ideal over $\hat{S}$. Let x, z ∈ $\hat{g}_t$ and y ∈ $\hat{S}$. So $\hat{g}_n(x) \geq t_n$ and $\hat{g}_n(z) \geq t_n$ for every n ∈ {1,2,...,m}. Due to the fact that $\hat{g}$ is a MPFGB-ideal, we obtain $\hat{g}_n((xy)z) \geq \hat{g}_n(x) \wedge \hat{g}_n(z) \geq t_n \wedge t_n = t_n$ for every n ∈ {1,2,...,m}. Thus (xy)z ∈ $\hat{g}_t$, that is $\hat{g}_t$ is a GB-ideal over $\hat{S}$.

Conversely, let $\hat{g}_t \neq \varphi$ is a GB-ideal over $\hat{S}$. On contrary considered that $\hat{g}$ is not a MPFGB-ideal over $\hat{S}$. Suppose x, y, z ∈ $\hat{S}$ with $\hat{g}_n((xy)z) < \hat{g}_n(x) \wedge \hat{g}_n(z)$ for any n ∈ {1,2,...,m}. Suppose $t_n = \hat{g}_n(x) \wedge \hat{g}_n(z)$ for every n ∈ {1,2,...,m}. Then x, z ∈ $\hat{g}_t$ but (xy)z ∉ $\hat{g}_t$, this contradicts the hypothesis. Hence $\hat{g}((xy)z) \geq \hat{g}(x) \wedge \hat{g}(z)$, that is $\hat{g}$ is a MPFGB-ideal over $\hat{S}$. Now, we define the multi-polar fuzzy bi-ideal (MPFB-ideal) over $\hat{S}$.

Definition 3.7 A multi-polar fuzzy sub LA-semigroup $\hat{g} = (\hat{g}_1,\hat{g}_2,...,\hat{g}_m)$ over $\hat{S}$ is a MPFB-ideal over $\hat{S}$ if for each x, y, z ∈ $\hat{S}$, $\hat{g}((xy)z) \geq \hat{g}(x) \wedge \hat{g}(z)$ that is, $\hat{g}_n((xy)z) \geq \hat{g}_n(x) \wedge \hat{g}_n(z)$ for each n ∈ {1,2,3,...,m}.

Lemma 3.7 A subset $\hat{H}$ over $\hat{S}$ which is non-empty is a bi-ideal over $\hat{S}$ iff $\hat{C}_{\hat{H}}$ is a MPFB-ideal over $\hat{S}$.

Proof. It is followed by Lemmas 3.2 and 3.5.

Lemma 3.8 A multi-polar fuzzy sub LA-semigroup $\hat{g}$ of $\hat{S}$ is a MPFB-ideal over $\hat{S}$ iff $(\hat{g} \circ \delta) \circ \hat{g} \leq \hat{g}$.

Proof. Follows from Lemma 3.6.

Proposition 3.3 Let $\hat{g} = (\hat{g}_1,\hat{g}_2,...,\hat{g}_m)$ is a MPF-sub LA-semigroup over $\hat{S}$. So $\hat{g}$ is a MPFB-ideal over $\hat{S}$ iff $\hat{g}_t = \{x \in \hat{S} \mid g(x) \geq t\} \neq \varphi$ is a bi-ideal over $\hat{S}$ for every $t = (t_1,t_2,t_3,...,t_m) \in (0,1]^m$.

Proof. It is followed by Proposition 3.2.

Remark 3.1 Every MPFB-ideal of $\hat{S}$ is a MPFGB-ideal over $\hat{S}$.

The example below illustrate that the converse may not hold.

Example 3.3 Let $\hat{S}$ = {p, q, r, s} be an LA-semigroup under binary operation "·" described below in Table 4.

Table 4. LA-semigroup

•	p	q	r	s
p	s	s	q	q
Q	s	s	s	s
R	s	s	q	s
S	s	s	s	s

Consider $\hat{g} = (\hat{g}_1,\hat{g}_2,\hat{g}_3,\hat{g}_4)$ be a 4-polar fuzzy subset over $\hat{S}$ with $\hat{g}(p) = (0.2,0.4,0.4,0.5)$, $\hat{g}(q) = (0,0,0,0)$, $\hat{g}(r) = (0,0,0,0)$, $\hat{g}(s) = (0.6,0.7,0.8,0.9)$. Thus it is simple to reveal that $\hat{g}$ is a 4-polar fuzzy generalized bi-ideal over $\hat{S}$. Now, $\hat{g}(q) = \hat{g}(p\cdot s) = (0,0,0,0) \ngeq (0.2,0.4,0.4,0.5) = \hat{g}(p) \wedge \hat{g}(s)$. So $\hat{g}$ is not a bi-ideal over $\hat{S}$.

Now we express the multi-polar fuzzy quasi-ideal (MPFQ-ideal) over $\hat{S}$.

Definition 3.8 A MPF-subset $\hat{g} = (\hat{g}_1,\hat{g}_2,...,\hat{g}_m)$ over $\hat{S}$ is a MPFQ-ideal over $\hat{S}$ if $(\hat{g} \circ \delta) \wedge (\delta \circ \hat{g}) \leq \hat{g}$, means that $(\hat{g}_n \circ \delta_n) \wedge (\delta_n \circ \hat{g}_n) \leq \hat{g}_n$ for every $n \in \{1,2,3,...,m\}$.

Lemma 3.9 A subset $\hat{J}$ over $\hat{S}$ which is non-empty is a quasi-ideal over $\hat{S}$ iff the multi-polar characteristic function $\hat{C}_{\hat{J}}$ of $\hat{J}$ is a MPFQ-ideal over $\hat{S}$.

Proof. Consider that $\hat{J}$ be a quasi-ideal over $\hat{S}$, i.e $\hat{J}\hat{S} \cap \hat{S}\hat{J} \subseteq \hat{J}$. We show that $(\hat{C}_{\hat{J}} \circ \delta) \wedge (\delta \circ \hat{C}_{\hat{J}}) \leq \hat{C}_{\hat{J}}$, means that

$((\hat{C}_{\hat{J}} \circ \delta) \wedge (\delta \circ \hat{C}_{\hat{J}}))(x) \leq \hat{C}_{\hat{J}}(x)$ for all $x \in \hat{S}$.

Let we have two cases,

Case1 : If $x \in \hat{J}$, then $\hat{C}_{\hat{J}}(x) = (1,1,...,1) \geq ((\hat{C}_{\hat{J}} \circ \delta) \wedge (\delta \circ \hat{C}_{\hat{J}}))(x)$.

Therefore $(\hat{C}_{\hat{J}} \circ \delta) \wedge (\delta \circ \hat{C}_{\hat{J}}) \leq \hat{C}_{\hat{J}}$.

Case 2 : If $x \notin \hat{J}$, so $x \notin \hat{J}\hat{S} \cap \hat{S}\hat{J}$. This implies that $x \neq ab$ or $x \neq cd$ for any $a \in \hat{J}$, $b \in \hat{S}$, $c \in \hat{S}$, $d \in \hat{J}$. Thus either $(\hat{C}_{\hat{J}} \circ \delta)(x) = (0,0,...,0)$ or $(\delta \circ \hat{C}_{\hat{J}})(x) = (0,0,...,0)$, means that $((\hat{C}_{\hat{J}} \circ \delta) \wedge (\delta \circ \hat{C}_{\hat{J}}))(x) = (0,0,...,0) \leq \hat{C}_{\hat{J}}(x)$. So that $(\hat{C}_{\hat{J}} \circ \delta) \wedge (\delta \circ \hat{C}_{\hat{J}}) \leq \hat{C}_{\hat{J}}$.

Conversely, let $z \in \hat{J}\hat{S} \cap \hat{S}\hat{J}$. Thus $z = ax$ and $z = yb$, where $x, y \in \hat{S}$ and $a, b \in \hat{J}$. Since $\hat{C}_{\hat{J}}$ is a MPFQ-ideal over $\hat{S}$, we get

$\hat{C}_{\hat{J}}(z) \geq ((\hat{C}_{\hat{J}} \circ \delta) \wedge (\delta \circ \hat{C}_{\hat{J}}))(z)$

$= (\hat{C}_{\hat{J}} \circ \delta)(z) \wedge (\delta \circ \hat{C}_{\hat{J}})(z)$

$= \{\bigvee_{z=uv}\{\hat{C}_{\hat{J}}(u) \wedge \delta(v)\}\} \wedge \{\bigvee_{z=pq}\{\delta(p) \wedge \hat{C}_{\hat{J}}(q)\}\}$

$\geq \{\hat{C}_{\hat{J}}(a) \wedge \delta(x)\} \wedge \{\delta(y) \wedge \hat{C}_{\hat{J}}(b)\}$

$= (1,1,...,1)$ since $z = ax$ and $z = yb$.

Thus $\hat{C}_{\hat{J}}(z) = (1,1,...,1)$. Hence $z \in \hat{J}$.

Proposition 3.4 Consider $\hat{g} = (\hat{g}_1,\hat{g}_2,...,\hat{g}_m)$ be a MPF-subset over $\hat{S}$. Thus $\hat{g}$ is a MPFQ-ideal over $\hat{S}$ iff $\hat{g}_t = \{s \in \hat{S} \mid \hat{g}(s) \geq t\} \neq \varphi$ is a quasi-ideal over $\hat{S}$ for every $t = (t_1,t_2,t_3,...,t_m) \in (0,1]^m$.

Proof. Consider $\hat{g}$ be a MPFQ-ideal over $\hat{S}$. To show that $\hat{g}_t\hat{S} \cap \hat{S}\hat{g}_t \subseteq \hat{g}_t$. Let $z \in \hat{g}_t\hat{S} \cap \hat{S}\hat{g}_t$. Then $z \in \hat{g}_t\hat{S}$ and $z \in \hat{S}\hat{g}_t$. So $z = ax$ and $z = yb$ for some $x,y \in \hat{S}$ and $a,b \in \hat{g}_t$. Thus $\hat{g}_n(a) \geq t_n$ and $\hat{g}_n(b) \geq t_n$ for every $n \in \{1,2,3,...,m\}$. Now,

$(\hat{g}_n \circ \delta_n)(z) = \bigvee_{z=uv}\{\hat{g}_n(u) \wedge \delta_n(v)\}$

$\geq \hat{g}_n(a) \wedge \delta_n(x)$ because $z = ax$

$= \hat{g}_n(a) \wedge 1$

$= \hat{g}_n(a)$

$\geq t_n$

So, $(\hat{g}_n \circ \delta_n)(z) \geq t_n$ for each $n \in \{1,2,...,m\}$. Now,

$(\delta_n \circ \hat{g}_n)(z) = \bigvee_{z=uv}\{\delta_n(u) \wedge \hat{g}_n(v)\}$

$\geq \delta_n(y) \wedge \hat{g}_n(b)$ because $z = yb$

$= 1 \wedge \hat{g}_n(b)$

$= \hat{g}_n(b)$

$\geq t_n$

So, $(\delta_n \circ \hat{g}_n)(z) \geq t_n$ for every $n \in \{1,2,...,m\}$.

Thus, $((\hat{g}_n \circ \delta_n) \wedge (\delta_n \circ \hat{g}_n))(z)$

$= ((\hat{g}_n \circ \delta_n)(z) \wedge (\delta_n \circ \hat{g}_n)(z) \geq t_n \wedge t_n = t_n$

for every $n \in \{1,2,3,...,m\}$. So, $((\hat{g} \circ \delta) \wedge (\delta \circ \hat{g}))(z) \geq t$. As $\hat{g}(z) \geq ((\hat{g} \circ \delta) \wedge (\delta \circ \hat{g}))(z) \geq t$, thus $z \in \hat{g}_t$. Therefore it is proved that $\hat{g}_t$ is a quasi-ideal over $\hat{S}$.

Conversely, on contrary, let $\hat{g}$ is not a MPFQ-ideal over $\hat{S}$. Let z ∈ $\hat{S}$ be such that $\hat{g}_n(z) < (\hat{g}_n \circ \delta_n)(z) \wedge (\delta_n \circ \hat{g}_n)(z)$ for any n ∈{1,2,...,m}. Take t_n ∈ (0,1] with $t_n = (\hat{g}_n \circ \delta_n)(z) \wedge (\delta_n \circ \hat{g}_n)(z)$ for every n ∈{1,2,3,...,m}. It follows that $z \in (\hat{g}_n \circ \delta_n)t_n$ and $z \in (\delta_n \circ \hat{g}_n)t_n$ but $z \notin (\hat{g}_n)t_n$ for some n. Therefore, $z \in (\hat{g} \circ \hat{S})t$ and $z \in (\hat{S} \circ \hat{g})_t$ but $z \notin \hat{g}_t$. Which leads to contradiction.

This proves that $(\hat{g} \circ \delta) \wedge (\delta \circ \hat{g}) \leq \hat{g}$.

Lemma 3.10 Every multi-polar fuzzy one-sided ideal over $\hat{S}$ is a MPFQ-ideal over $\hat{S}$.

Proof. It is followed by Lemma 3.3.

The subsequent example demonstrates that the converse may not hold.

Example 3.4 Let $\hat{S}$ = {r, s, t, u} be an LA-semigroup under binary operation "·" described below in Table 5.

Table 5. LA-semigroup

•	r	s	t	U
R	r	s	t	U
S	u	t	t	T
T	t	t	t	T
U	s	t	t	T

Define a 5-polar fuzzy subset $\hat{g} = (\hat{g}_1,\hat{g}_2,\hat{g}_3,\hat{g}_4,\hat{g}_5)$ of $\hat{S}$ as follows:

$\hat{g}(s) = \hat{g}(t) = (0.4,0.4,0.5,0.5,0.6)$, $\hat{g}(r) = \hat{g}(u) = (0,0,0,0,0)$. Thus it is simple to reveal that $\hat{g}_t$ is a quasi-ideal over $\hat{S}$. Therefore by using Proposition 4, $\hat{g}$ is a 5-polar FQ-ideal over Ŝ. Now,

$$\hat{g}(u) = \hat{g}(s.r) = (0,0,0,0,0)$$

$$\ngeq (0.4,0.4,0.5,0.5,0.6) = \hat{g}(s).$$

So $\hat{g}$ is not a 5-polar FR-ideal over $\hat{S}$.

Lemma 3.11 Suppose that $\hat{g} = (\hat{g}_1,\hat{g}_2,...,\hat{g}_m)$ and $\hat{h} = (\hat{h}_1,\hat{h}_2,...,\hat{h}_m)$ be MPFR-ideal and MPFL-ideal over $\hat{S}$. Then $\hat{g} \wedge \hat{h}$ is a multi-polar FQ-ideal over $\hat{S}$.

Proof. Consider $\hat{g} = (\hat{g}_1,\hat{g}_2,...,\hat{g}_m)$ and $\hat{h} = (\hat{h}_1,\hat{h}_2,...,\hat{h}_m)$ be MPFR-ideal and MPFL-ideal over $\hat{S}$. Let s ∈ $\hat{S}$. If s ≠ ab for a,b ∈ $\hat{S}$. We have

$$((\hat{g} \wedge \hat{h})\circ\delta) \wedge (\delta \circ (\hat{g} \wedge \hat{h})) \leq (\hat{g} \wedge \hat{h}).$$

If s = pq for p,q ∈ $\hat{S}$, then

$$(((\hat{g}_n \wedge \hat{h}_n) \circ \delta_n) \wedge (\delta_n \circ (\hat{g}_n \wedge \hat{h}_n)))(s)$$

$$= ((\hat{g}_n \wedge \hat{h}_n) \circ \delta_n)(s) \wedge (\delta_n \circ (\hat{g}_n \wedge \hat{h}_n))(s)$$

$$= \begin{Bmatrix} \bigvee_{s=pq}\{(\hat{g}_n \wedge \hat{h}_n)(p) \wedge \delta_n(q)\} \wedge \\ \bigvee_{s=pq}\{\delta_n(p) \wedge (\hat{g}_n \wedge \hat{h}_n)(q)\} \end{Bmatrix}$$

$$= \bigvee_{s=pq}\{(\hat{g}_n \wedge \hat{h}_n)(p)\} \wedge \bigvee_{s=pq}\{(\hat{g}_n \wedge \hat{h}_n)(q)\}$$

$$= \bigvee_{s=pq}\{(\hat{g}_n \wedge \hat{h}_n)(p) \wedge (\hat{g}_n \wedge \hat{h}_n)(q)\}$$

$$= \bigvee_{s=pq}\{(\hat{g}_n(p) \wedge \hat{h}_n(p)) \wedge (\hat{g}_n(q) \wedge \hat{h}_n(q))$$

$$\leq \bigvee_{s=pq}\{\hat{g}_n(p) \wedge \hat{h}_n(q)\}$$

$$\leq \bigvee_{s=pq}\{(\hat{g}_n(pq) \wedge \hat{h}_n(pq)\}$$

$$= \bigvee_{s=pq}\{(\hat{g}_n \wedge \hat{h}_n)(pq)\}$$

$$= (\hat{g}_n \wedge \hat{h}_n)(s) \text{ for every } n \in \{1,2,...,m\}.$$

Thus $((\hat{g}\wedge\hat{h})\circ\delta) \wedge (\delta\circ(\hat{g}\wedge\hat{h})) \leq (\hat{g}\wedge\hat{h})$, that is $\hat{g} \wedge \hat{h}$ be a MPFQ-ideal over $\hat{S}$.

Now, we define the multi-polar fuzzy interior-ideal (MPFI-ideal) over $\hat{S}$.

Definition 3.9 A multi-polar fuzzy sub LA-semigroup $\hat{g} = (\hat{g}_1,\hat{g}_2,...,\hat{g}_m)$ of $\hat{S}$ is a MPFI-ideal over $\hat{S}$ if for each x, a, y ∈ $\hat{S}$, $\hat{g}((xa)y) \geq \hat{g}(a)$, that is $\hat{g}_n((xa)y) \geq \hat{g}_n(a)$ for every n ∈ {1,2,3,...,m}.

Lemma 3.12 A subset $\hat{I}$ over $\hat{S}$ which is non-empty is an interior ideal over $\hat{S}$ iff the multi-polar characteristic function $\hat{C}_{\hat{I}}$ over $\hat{I}$ is a MPFI-ideal over $\hat{S}$.

Proof: Consider that $\hat{I}$ is an interior ideal over $\hat{S}$. From Lemma 2, $\hat{C}_{\hat{I}}$ is a multi-polar fuzzy sub LA-semigroup over $\hat{S}$. Now, we show that $\hat{C}_{\hat{I}}((pq)r) \geq \hat{C}_{\hat{I}}(q)$ for every p,q,r ∈ $\hat{S}$. Let we have the four cases,

Case 1 : Consider that q ∈ $\hat{I}$ and p, r ∈ $\hat{S}$. Then $\hat{C}_{\hat{I}}(q) = (1,1,...,1)$. Since $\hat{I}$ is an interior ideal over $\hat{S}$, so (pq)r ∈ $\hat{I}$. Then $\hat{C}_{\hat{I}}((pq)r) = (1,1,...,1)$. Hence $\hat{C}_{\hat{I}}((pq)r) \geq \hat{C}_{\hat{I}}(q)$.

Case 2 : Let $q \notin \hat{I}$ and $p,r \in \hat{S}$. Then $\hat{C}_{\hat{I}}(q) = (0,0,...,0)$. Clearly, $\hat{C}_{\hat{I}}((pq)r) \geq \hat{C}_{\hat{I}}(q)$. Hence the multi-polar characteristic function $\hat{C}_{\hat{I}}$ over $\hat{I}$ is an multi-polar FI-ideal over $\hat{S}$.

Conversely, consider that $\hat{C}_{\hat{I}}$ is a MPFI-ideal over $\hat{S}$. Then by Lemma 2, $\hat{I}$ is a sub LA-semigroup over $\hat{S}$. Let $p,r \in \hat{S}$ and $q \in \hat{I}$. Then, $\hat{C}_{\hat{I}}(q) = (1,1,...,1)$. By the hypothesis, $\hat{C}_{\hat{I}}((pq)r) \geq \hat{C}_{\hat{I}}(q) = (1,1,...,1)$. Hence $\hat{C}_{\hat{I}}((pq)r) = (1,1,...,1)$. This proves that $(pq)r \in \hat{I}$, that is $\hat{I}$ is an interior ideal over $\hat{S}$.

Lemma 3.13 Let $\hat{g}$ be a MPF-sub LA-semigroup over $\hat{S}$. Then $\hat{g}$ is a MPFI-ideal over $\hat{S}$ iff $(\delta \circ \hat{g}) \circ \delta \leq \hat{g}$.

Proof. Let $\hat{g} = (\hat{g}_1,\hat{g}_2,...,\hat{g}_m)$ be a multi-polar FI-ideal over $\hat{S}$. We demonstrate that $(\delta \circ \hat{g}) \circ \delta \leq \hat{g}$. Let $z \in \hat{S}$. Then for every $n \in \{1,2,...,m\}$.

$$((\delta_n \circ \hat{g}_n) \circ \delta_n)(z) = \bigvee_{z=uv}\{(\delta_n \circ \hat{g}_n)(u) \wedge \delta_n(v)\}$$

$$= \bigvee_{z=uv}\{(\delta_n \circ \hat{g}_n)(u)\}$$

$$= \bigvee_{z=uv}\{\bigvee_{u=ab}\{\delta_n(a) \wedge \hat{g}_n(b)\}$$

$$= \bigvee_{z=uv}\{\bigvee_{u=ab}\{\hat{g}_n(b)\}$$

$$= \bigvee_{z=(ab)v}\{\hat{g}_n(b)\}$$

$$\leq \bigvee_{z=(ab)v}\{\hat{g}_n((ab)v)\}$$

$$= \hat{g}_n(z) \text{ for every } n \in \{1,2,...,m\}.$$

Thus $(\delta \circ \hat{g}) \circ \delta \leq \hat{g}$.

In the reverse, assume that $(\delta \circ \hat{g}) \circ \delta \leq \hat{g}$. We only prove that $\hat{g}_n((xa)y) \geq \hat{g}_n(a)$ for each $x,a,y \in \hat{S}$ and for every $n \in \{1,2,...,m\}$. Let $z = (xa)y$. Now for every $n \in \{1,2,...,m\}$.

$$\hat{g}_n((xa)y) \geq ((\delta_n \circ \hat{g}_n) \circ \delta_n)((xa)y)$$

$$= \bigvee_{(xa)y=uv}\{(\delta_n \circ \hat{g}_n)(u) \wedge \delta_n(v)\}$$

$$\geq (\delta_n \circ \hat{g}_n)(xa) \wedge \delta_n(y)$$

$$= (\delta_n \circ \hat{g}_n)(xa)$$

$$= \bigvee_{xa=pq}\{(\delta_n(p) \wedge \hat{g}_n(q)\}$$

$$\geq \delta_n(x) \wedge \hat{g}_n(a)$$

$$= \hat{g}_n(a) \text{ for all } n \in \{1,2,...,m\}.$$

So, $\hat{g}_n((xa)y) \geq \hat{g}_n(a)$ for each $n \in \{1,2,...,m\}$. Thus $\hat{g}$ is a MPFI-ideal over Ŝ.

Proposition 3.5 Consider $\hat{g} = (\hat{g}_1,\hat{g}_2,...,\hat{g}_m)$ be a MPF-subset over Ŝ. Then $\hat{g}$ is a multi-polar FI-ideal over $\hat{S}$ iff $\hat{g}_t = \{x \in \hat{S} \mid \hat{g}(x) \geq t\} \neq \varphi$ is an interior ideal over Ŝ for each $t = (t_1,t_2,t_3,...,t_m) \in (0,1]^m$.

Proof. It can be proved on the same lines of Propositions 3.1 and 3.2.

4. REGULAR LA-SEMIGROUPS CHARACTERIZED BY MULTI-POLAR FUZZY IDEALS

Definition 4.1 If for every element s in the LA-semigroup $\hat{S}$, there exists $r \in \hat{S}$ such that s can be expressed as $s = (sr)s$ then $\hat{S}$ is a regular LA-semigroup.

Theorem 4.1 [15] Let $\hat{S}$ possesses e with $(ae)\hat{S} = a\hat{S}$ for each $a \in \hat{S}$. So the subsequent assertions are equivalent.

(1) $\hat{S}$ is regular

(2) For all R-ideal $\hat{R}$ and L-ideal $\hat{L}$ over $\hat{S}$ we have $\hat{R} \cap \hat{L} = \hat{R}\hat{L}$.

(3) $\hat{J} = (\hat{J}\hat{S})\hat{J}$ for all Q-ideal $\hat{J}$ over $\hat{S}$.

Theorem 4.2 If $\hat{S}$ possesses e with $(re)\hat{S} = r\hat{S}$ for each $r \in \hat{S}$. Then any MPFQ-ideal over $\hat{S}$ is a MPFB-ideal over $\hat{S}$.

Proof. Consider $\hat{g} = (\hat{g}_1,\hat{g}_2,...,\hat{g}_m)$ be any MPFQ-ideal over $\hat{S}$. Take $p,q \in \hat{S}$. Then,

$$\hat{g}_n(pq) \geq ((\hat{g}_n \circ \delta_n) \wedge (\delta_n \circ \hat{g}_n))(pq)$$

$$= (\hat{g}_n \circ \delta_n)(pq) \wedge (\delta_n \circ \hat{g}_n)(pq)$$

$$= \begin{Bmatrix} \bigvee_{pq=ab}\{\hat{g}_n(a) \wedge \delta_n(b)\} \wedge \\ \bigvee_{pq=uv}\{\delta_n(u) \wedge \hat{g}_n(v)\} \end{Bmatrix}$$

$$\geq \{\hat{g}_n(p) \wedge \delta_n(q)\} \wedge \{\delta_n(p) \wedge \hat{g}_n(q)\}$$

$$= \{\hat{g}_n(p) \wedge 1\} \wedge \{1 \wedge \hat{g}_n(q)\}$$

$$= \hat{g}_n(p) \wedge \hat{g}_n(q) \text{ for all } n \in \{1,2,...,m\}.$$

So, $\hat{g}(pq) \geq \hat{g}(p) \wedge \hat{g}(q)$.

Now, let $p,q,r \in \hat{S}$. Then,

$$(\delta_n \circ \hat{g}_n)((pq)r) = \bigvee_{(pq)r=uv}\{\delta_n(u) \wedge \hat{g}_n(v)\}$$

$$\geq \delta_n(pq) \wedge \hat{g}_n(r)$$

$$= 1 \wedge \hat{g}_n(r)$$

$$= \hat{g}_n(r)$$

So, $(\delta_n \circ \hat{g}_n)((pq)r) \geq \hat{g}_n(r)$ for all n ∈ {1,2,...,m}.

Since $(pq)r = (pq)(er) = (pe)(qr) \in (pe)\hat{S} = p\hat{S}$, so $(pq)r = ps$ for some s ∈ $\hat{S}$. Thus,

$(\hat{g}_n \circ \delta_n)((pq)r) = \vee_{(pq)r=ab}\{\hat{g}_n(a) \wedge \delta_n(b)\}$

$\geq \hat{g}_n(p) \wedge \delta_n(s)$ since (pq)r = ps

$= \hat{g}_n(p) \wedge 1$

$= \hat{g}_n(p)$

So, $(\hat{g}_n \circ \delta_n)((pq)r) \geq \hat{g}_n(p)$ for every n ∈ {1,2,...,m}.

Now, by our assumption

$\hat{g}_n((pq)r) \geq ((\hat{g}_n \circ \delta_n) \wedge (\delta_n \circ \hat{g}_n))((pq)r)$

$= (\hat{g}_n \circ \delta_n)((pq)r) \wedge (\delta_n \circ \hat{g}_n)((pq)r)$

$\geq \hat{g}_n(p) \wedge \hat{g}_n(r)$ for every n ∈ {1,2,...,m}.

Thus, $\hat{g}((pq)r) \geq \hat{g}(p) \wedge \hat{g}(r)$. This proves that $\hat{g}$ is an MPFB-ideal over $\hat{S}$.

Theorem 4.3 The subsequent statements are equivalent for an LA-semigroup $\hat{S}$.

(1) $\hat{S}$ is regular

(2) $\hat{g} \wedge \hat{h} = \hat{g} \circ \hat{h}$ for any MPFR-ideal $\hat{g}$ and MPFL-ideal $\hat{h}$ over $\hat{S}$.

Proof. (1) ⇒ (2): Consider $\hat{g} = (\hat{g}_1, \hat{g}_2, ..., \hat{g}_m)$ and $\hat{h} = (\hat{h}_1, \hat{h}_2, ..., \hat{h}_m)$ be any MPFR-ideal and MPFL-ideal of $\hat{S}$. Let a ∈ $\hat{S}$, we get

$(\hat{g}_n \circ \hat{h}_n)(a) = \vee_{a=yz}\{\hat{g}_n(y) \wedge \hat{h}_n(z)\}$

$\leq \vee_{a=yz}\{\hat{g}_n(yz) \wedge \hat{h}_n(yz)\}$

$= \hat{g}_n(a) \wedge \hat{h}_n(a)$

$= (\hat{g}_n \wedge \hat{h}_n)(a)$ for all n ∈ {1,2,...,m}.

So, $(\hat{g} \circ \hat{h}) \leq (\hat{g} \wedge \hat{h})$.

By assertion (1), for each a ∈ $\hat{S}$, we have a = (ax)a for some x ∈ $\hat{S}$. So we get

$(\hat{g}_n \wedge \hat{h}_n)(a) = \hat{g}_n(a) \wedge \hat{h}_n(a)$

$\leq \hat{g}_n(ax) \wedge \hat{h}_n(a)$

$\leq \vee_{a=yz}\{\hat{g}_n(y) \wedge \hat{h}_n(z)\}$

$= (\hat{g}_n \circ \hat{h}_n)(a)$ for all n ∈ {1,2,...,m}.

Thus, $(\hat{g} \circ \hat{h}) \geq (\hat{g} \wedge \hat{h})$. Hence proved that $(\hat{g} \wedge \hat{h}) = (\hat{g} \circ \hat{h})$.

(2) ⇒ (1): Suppose that a ∈ $\hat{S}$. Thus a$\hat{S}$ is a L-ideal over $\hat{S}$ and a$\hat{S}$∪$\hat{S}$a is a R-ideal over $\hat{S}$ generated by a say a$\hat{S} = \hat{L}$ and a$\hat{S}$∪$\hat{S}$a = $\hat{R}$. Now $\hat{C}_{\hat{L}}$ and $\hat{C}_{\hat{R}}$ the multi-polar characteristic functions of $\hat{L}$ and $\hat{R}$ are MPFL-ideal and MPFR-ideal over $\hat{S}$ by using Lemma 3.2. Hence, from Lemma 3.1 and assertion (2) we get

$\hat{C}_{\hat{R}\hat{L}} = (\hat{C}_{\hat{R}} \circ \hat{C}_{\hat{L}})$ from Lemma 3.1

$= (\hat{C}_{\hat{R}} \wedge \hat{C}_{\hat{L}})$ from 2

$= \hat{C}_{\hat{R} \cap \hat{L}}$ by Lemma 3.1.

This proves that $\hat{R} \cap \hat{L} = \hat{R}\hat{L}$. Thus $\hat{S}$ is regular from Theorem 4.1.

Theorem 4.4 Consider e ∈ $\hat{S}$ with (ae)$\hat{S}$ = a$\hat{S}$ for each a ∈ $\hat{S}$. Thus the subsequent assertions are equivalent.

(1) $\hat{S}$ is regular

(2) $\hat{g} = (\hat{g} \circ \delta) \circ \hat{g}$ for any MPFGB-ideal $\hat{g}$ over $\hat{S}$.

(3) $\hat{g} = (\hat{g} \circ \delta) \circ \hat{g}$ for each MPFB-ideal $\hat{g}$ over $\hat{S}$.

Proof. (1) ⇒ (2): Consider $\hat{g} = (\hat{g}_1, \hat{g}_2, ..., \hat{g}_m)$ be a MPFGB-ideal over $\hat{S}$. Let a ∈ $\hat{S}$, so by assertion (1), a = (ax)a for some x ∈ $\hat{S}$. So, we get

$((\hat{g}_n \circ \delta_n) \circ \hat{g}_n)(a)$

$= \vee_{a=yz}\{(\hat{g}_n \circ \delta_n)(y) \wedge \hat{g}_n(z)\}$ for some y, z ∈ $\hat{S}$

$\geq (\hat{g}_n \circ \delta_n)(ax) \wedge \hat{g}_n(a)$ since a = (ax)a

$= \vee_{ax=pq}\{\hat{g}_n(p) \wedge \delta_n(q)\} \wedge \hat{g}_n(a)$

$\geq \{\hat{g}_n(a) \wedge \delta_n(x)\} \wedge \hat{g}_n(a)$

$= \hat{g}_n(a)$ for all n ∈ {1,2,...,m}.

Hence proved that $(c \circ \delta) \circ \hat{g} \geq \hat{g}$.

Because $\hat{g}$ is a MPFGB-ideal over $\hat{S}$. Thus, we get

$((\hat{g}_n \circ \delta_n) \circ \hat{g}_n)(a)$

$= \vee_{a=yz}\{(\hat{g}_n \circ \delta_n)(y) \wedge \hat{g}_n(z)\}$ for some y,z ∈ $\hat{S}$
$= \vee_{a=yz}\{\vee_{y=pq}\{\hat{g}_n(p) \wedge \delta_n(q)\} \wedge \hat{g}_n(z)\}$ for p,q ∈ $\hat{S}$

$= \bigvee_{a=yz}\{\bigvee_{y=pq}\{\hat{g}_n(p) \wedge \hat{g}_n(z)\}\}$

$\leq \bigvee_{a=yz}\{\bigvee_{y=pq}\{\hat{g}_n((pq)z)\}\}$

$= \bigvee_{a=yz}\{\hat{g}_n(yz)\}$

$= \hat{g}_n(a)$ for all n ∈ {1,2,...,m}.

So, $(\hat{g} \circ \delta) \circ \hat{g} \leq \hat{g}$. Thus $\hat{g} = (\hat{g} \circ \delta) \circ \hat{g}$.

(2) ⇒ (3): It is straightforward.

(3) ⇒ (1): Consider $\hat{J}$ be any quasi-ideal over $\hat{S}$. Since $(\hat{J}\hat{S})\hat{J} \subseteq (\hat{J}\hat{S})\hat{S} = (\hat{J}\hat{S})(e\hat{S}) = (\hat{J}e)(\hat{S}\hat{S}) = (\hat{J}e)\hat{S} = \hat{J}\hat{S}$ and $(\hat{J}\hat{S})\hat{J} \subseteq (\hat{S}\hat{S})\hat{J} = \hat{S}\hat{J}$. Therefore $(\hat{J}\hat{S})\hat{J} \subseteq \hat{J}\hat{S} \cap \hat{S}\hat{J} \subseteq \hat{J}$.

Now, let a ∈ $\hat{J}$ such that a = yz for some y, z ∈ $\hat{S}$. Since by Lemma 3.9, $\hat{C}_J$ is a MPFQ-ideal over $\hat{S}$. Therefore $\hat{C}_J$ is an MPFB-ideal over $\hat{S}$ by Theorem 4.2. Thus, we get

$((\hat{C}_J \circ \delta) \circ \hat{C}_J)(a) = \hat{C}_J(a)$ by using condition (3)

$= (1,1,...,1)$

Hence $((\hat{C}_J \circ \delta) \circ \hat{C}_J)(a) = (1,1,...,1)$. So, there are elements u, v ∈ $\hat{S}$ so that $(\hat{C}_J \circ \delta)(u) = (1,1,...,1)$ and $\hat{C}_J(v) = (1,1,...,1)$ with a = uv. Since $(\hat{C}_J \circ \delta)(u) = (1,1,...,1)$. So there are elements w, e ∈ $\hat{S}$ such that $\hat{C}_J(w) = (1,1,...,1)$ and δ(e) = (1,1,...,1) with u = we. Thus w, v ∈ $\hat{J}$ and e ∈ $\hat{S}$ and so a = uv = (we)v ∈ $(\hat{J}\hat{S})\hat{J}$. Hence $\hat{J} \subseteq (\hat{J}\hat{S})\hat{J}$. So, $\hat{J} = (\hat{J}\hat{S})\hat{J}$. Thus $\hat{S}$ is regular from Theorem 4.1.

Theorem 4.5 Consider e ∈ $\hat{S}$ with $(ae)\hat{S} = a\hat{S}$ for each a ∈ $\hat{S}$. Thus the subsequent statements are equivalent.

(1) $\hat{S}$ is regular

(2) Consider any MPFR-ideal $\hat{g}$, any MPFGB-ideal $\hat{h}$, and any MPFL-ideal $\hat{I}$ over $\hat{S}$, this

$(\hat{g} \circ \hat{h}) \circ \hat{I} \geq (\hat{g} \wedge \hat{h}) \wedge \hat{I}$ holds.

(3) Consider any MPFR-ideal $\hat{g}$, any MPFB-ideal $\hat{h}$, and any MPFL-ideal $\hat{I}$ over $\hat{S}$, this

$(\hat{g} \circ \hat{h}) \circ \hat{I} \geq (\hat{g} \wedge \hat{h}) \wedge \hat{I}$ holds.

(4) Consider any MPFR-ideal $\hat{g}$, any MPFQ-ideal $\hat{h}$, and any MPFL-ideal $\hat{I}$ of $\hat{S}$, this

$(\hat{g} \circ \hat{h}) \circ \hat{I} \geq (\hat{g} \wedge \hat{h}) \wedge \hat{I}$ holds.

Proof. (1) ⇒ (2): Consider $\hat{g} = (\hat{g}_1,\hat{g}_2,...,\hat{g}_m)$, $\hat{h} = (\hat{h}_1,\hat{h}_2,...,\hat{h}_m)$, and $\hat{I} = (\hat{I}_1,\hat{I}_2,...,\hat{I}_m)$ be any MPFR-ideal, MPFGB-ideal and MPFL-ideal $\hat{I}$ over $\hat{S}$, respectively. Suppose that a ∈ $\hat{S}$, so by assertion (1) a = (ar)a for some r ∈ $\hat{S}$. It follows that, a = (ar)a = (ar)(ea) = (ae)(ra) = a(ra) since $(ae)\hat{S} = a\hat{S}$ for each a ∈ $\hat{S}$. Hence we get

$((\hat{g} \circ \hat{h}) \circ \hat{I})(a) = \bigvee_{a=uv}\{(\hat{g} \circ \hat{h})(u) \wedge \hat{I}(v)\}$

$\geq (\hat{g} \circ h)(a) \wedge \hat{I}(ra)$ as a = a(ra)

$\geq \bigvee_{a=pq}\{\hat{g}(p) \wedge \hat{h}(q)\} \wedge \hat{I}(a)$

$\geq (\hat{g}(ar) \wedge \hat{h}(a)) \wedge \hat{I}(a)$ as a = (ar)a

$\geq (\hat{g}(a) \wedge \hat{h}(a)) \wedge \hat{I}(a)$

$= ((\hat{g} \wedge \hat{h})(a)) \wedge \hat{I}(a)$

$= ((\hat{g} \wedge \hat{h}) \wedge \hat{I})(a)$

Hence proved that $(\hat{g} \circ \hat{h}) \circ \hat{I} \geq (\hat{g} \wedge \hat{h}) \wedge \hat{I}$.

(2) ⇒ (3) ⇒ (4): These are straight forward.

(4) ⇒ (1): Consider $\hat{g} = (\hat{g}_1,\hat{g}_2,...,\hat{g}_m)$ and $\hat{I} = (\hat{I}_1,\hat{I}_2,...,\hat{I}_m)$ be any MPFR-ideal and MPFL-ideal over $\hat{S}$. As δ be a MPFQ-ideal over $\hat{S}$, by the supposition, we get

$(\hat{g} \wedge \hat{I})(a) = ((\hat{g} \wedge \delta) \wedge \hat{I})(a)$

$\leq ((\hat{g} \circ \delta) \circ \hat{I})(a)$

$= \bigvee_{a=pq}\{(\hat{g} \circ \delta)(p) \wedge \hat{I}(q)\}$

$= \bigvee_{a=pq}\{(\bigvee_{p=uv}\{\hat{g}(u) \wedge \delta(v)\}) \wedge \hat{I}(q)\}$

$= \bigvee_{a=pq}\{(\bigvee_{p=uv}\{\hat{g}(u) \wedge 1\}) \wedge \hat{I}(q)\}$

$= \bigvee_{a=pq}\{(\bigvee_{p=uv}\hat{g}(u)) \wedge \hat{I}(q)\}$

$\leq \bigvee_{a=pq}\{(\bigvee_{p=uv}\{\hat{g}(uv)\}) \wedge \hat{I}(q)\}$

$= \bigvee_{a=pq}\{\hat{g}(p) \wedge \hat{I}(q)\}$

$= (\hat{g} \circ \hat{I})(a)$

Thus $(\hat{g} \circ \hat{I}) \geq (\hat{g} \wedge \hat{I})$ for any MPFR-ideal $\hat{g}$ and any MPFL-ideal $\hat{I}$ over $\hat{S}$. But $(\hat{g} \circ \hat{I}) \leq (\hat{g} \wedge \hat{I})$. This gives $(\hat{g} \circ \hat{I}) = (\hat{g} \wedge \hat{I})$. Thus $\hat{S}$ is regular by Theorem 4.3.

5. CONCLUSION

In this research paper, we have put forward the idea of MPF-sets which is an expansion of BPF-sets. Infact, the BPF-sets are useful mathematical model to demonstrate the positivity and negativity of goods. In this study we have examined the multi-information about the given data by defining the multi-polar fuzzy sets in LA-semigroups. Mainly, we have confined our attention to investigate how we can generalize the results of BPF-sets in terms of multi-polar fuzzy sets. Also detailed exposition of multi-polar fuzzy ideals in $\hat{S}$ have been studied. Moreover, this study can be used as a design for aggregation or classification and to define multi-valued relations. One such structure is the Pythagorean MPF-set which is hybrid model of both PFS and MPF-sets is presented by Naeem *et al.* [17]. Another related model is the Pythagorean MPF-sets, which was proposed by Riaz *et al.* [18]. The interval [0,1] is the range of a membership function, which illustrates a fuzzy set (F-set). A membership degree serves as an illustration of how individuals of a set are related.

6. CONFLICT OF INTEREST

The authors declare no conflict of interest.

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