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Research Article

Mathematical Analysis of Conducting and Dielectric Sphere in Fractional Space

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Abstract: This paper presents an analytical analysis of a sphere placed in fractional dimensional space. The Laplacian Equation in fractional space describes physics as a complex phenomenon. The general solution of the Laplacian equation in fractional space is obtained by the separable variable technique. We have investigated a close form solution for conducting sphere and dielectric sphere. Further, the electric potential and charge density, induced due to a point charge is calculated in fractional space, and also the energy radiated by the sphere is determined. The results are compared with the classical results by setting the fractional parameter α = 3 which normally lies in the limit $2 < \alpha \leq 3$.

1. INTRODUCTION

The idea of fractional dimensional space (FD Space) is very useful in the various branches of physics and it has been discussed by many researchers . Various scientists have applied it accordingly as Wilson [3] has discussed quantum field theory in fractional space. Further, the fractional space can be used as a parameter in the Ising limit of quantum field theory [6]. Stillinger [4] has given a brief introduction to this theme for the formulation of Schrodinger and Gibbsian statistical mechanics in the fractional space. Svozil and Zeilinger [10] have investigated operationalistic meanings of the dimension of space time that provides the possibility of predicted space time dimension. It is already stated that the fractional space time is less than 4. In the new era, Gauss law [11] has been formulated in the FD space. The solutions of electrostatic problems , have also been investigated in the fractional space for.

In this paper, we have focused on the problem of a sphere in an electrostatic field [16] and worked it out for fractional space. Some researchers [17] have also discussed this problem in fraction dimensional space. But we have calculated the electric field and energy in fractional dimensional space. We have also calculated its energy, charge density, dipole moment and electric field in fractional space for the outside sphere as well as inside the sphere. We have considered here both cases of the sphere "conducting and dielectric" in the electrostatic field. We have solved this problem analytically. First, we consider a conducting sphere, then we solve it for a dielectric sphere and finally, we solve for an electric field and energy radiated by the sphere. For the integer order , the original classical solution can be recovered.

2. MATHEMATICAL MODEL

Let us consider a conducting sphere (as shown in Figure 1) having radius $\boldsymbol{r_1}$ embedded in the host medium of permittivity $\boldsymbol{\epsilon_2}$ [16]. The point charge q is situated on $\boldsymbol{z-axis}$ at $\boldsymbol{z=\zeta>r_1}$. In FD space, we will find the potential on the sphere as well as the charge distribution. At any point outside the sphere, the entire potential is given as $\boldsymbol{\phi=\phi_0+\phi_1}$, where $\boldsymbol{\phi_0}$ is known as the potential of the outer source q and $\boldsymbol{\phi_1}$ indicates the potential of the induced charge distribution on the sphere. This potential is single valued, as in Stratton [16], and can be expressed as:

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$$\boldsymbol{\phi_1} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\boldsymbol{A_{nm}r^n} + \frac{\boldsymbol{B_{nm}}}{r^{n+1}}\right)\boldsymbol{P_n^m(cos\theta)exp\Im\psi} \quad (1)$$

where the unknown arbitrary constants are $\boldsymbol{A_{nm}}$ and $\boldsymbol{B_{nm}}$. But $\boldsymbol{\phi_1}$ must be analytic at infinity, as we set $\boldsymbol{A_{nm} = 0}$. Further, the potential $\boldsymbol{\phi_0}$ is primary and symmetric about $\boldsymbol{z - axis}$, here $\boldsymbol{m = 0}$, in this case. Therefore, the potential of the induced charge distribution on the sphere can be expressed as:

$$\boldsymbol{\phi_1} = \sum_{n=0}^{\infty}\frac{\boldsymbol{B_n}}{r^{n+1}}\boldsymbol{P_n(cos\theta)} \quad (2)$$

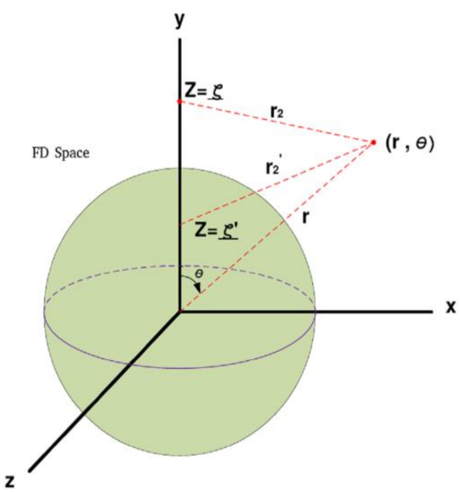

Fig. 1. A sphere placed in fractional dimensional space

In fractional space we can express as:

$$\phi_1 = \sum_{l=0}^{\infty}\frac{B_l}{r^{l+\alpha-2}}C_l^{\alpha/2-1}(cos\theta), (2 < \alpha \leq 3) \quad (3)$$

When $r < \zeta$, the expansion of the primary potential ϕ_0 can be expressed as:

$$\phi_0 = \frac{1}{4\pi\epsilon_2}\frac{q}{r_2} = \frac{1}{4\pi\epsilon}\frac{q}{\zeta}\sum_{l=0}^{\infty}\left(\frac{r}{\zeta}\right)^l C_l^{\alpha/2-1}(cos\theta), (\alpha \leq 3) \quad (4)$$

The resultant potential at the surface of sphere $r = r_1$ is:

$$\phi(r_1, \theta) = \phi_s = \sum_{l=0}^{\infty}\left[\frac{1}{4\pi\epsilon_2}\frac{q}{\zeta}\left(\frac{r_1}{\zeta}\right)^l + \frac{B_l}{r_1^{l+\alpha-2}}\right]C_l^{\alpha/2-1}(cos\theta), (2 < \alpha \leq 3) \quad (5)$$

As ϕ_s is constant and since the above equation must be valid for all values of θ, it means that the coefficients of $C_l^{\alpha/2-1}(cos\theta)$ must goes to zero for all values of l > 0. Here, the unknown coefficient b_n can be determined from the following set of relations:

$$b_0 = r_1\phi_s - \frac{q}{4\pi\epsilon_2}\frac{r_1}{\zeta} \quad (6)$$

$$B_l = \frac{-q}{4\pi\epsilon_2}\frac{qr_1^{2l+\alpha-2}}{\zeta^{l+1}}, (l > 0) \quad (7)$$

At any point, the potential outside the sphere is:

$$\phi = \frac{r_1\phi_s}{r} + \frac{1}{4\pi\epsilon_2}\frac{q}{r_2} - \frac{q}{4\pi\epsilon_2}\sum_{l=0}^{\infty}\frac{r_1^{2l+\alpha-2}}{\zeta^{l+1}}\frac{C_l^{\alpha/2-1}(\cos\theta)}{r^{l+\alpha-2}}, (2 < \alpha \leq 3) \quad (8)$$

To find the charge density [16], we can calculate the normal derivatives on the surface of the sphere as:

$$\left(\frac{\partial\phi}{\partial r}\right)_{r=r_1} = \frac{-\phi_s}{r_1} + \frac{q}{4\pi\epsilon_2}\sum_{l=0}^{\infty}(l + \alpha - 2)\frac{r_1^{l-1}}{\zeta^{l+1}}C_l^{\alpha/2-1}(cos\theta), (2 < \alpha \leq 3) \quad (9)$$

and the induced charge density can be calculated as:

$$\omega = -\epsilon_2\left(\frac{\partial\phi}{\partial r}\right)_{r=r_1} = \epsilon_2\frac{\phi_s}{r_1} - \frac{q}{4\pi}\sum_{l=0}^{\infty}(l + \alpha - 2)\frac{r_1^{l-1}}{\zeta^{l+1}}C_l^{\alpha/2-1}(cos\theta), (2 < \alpha \leq 3) \quad (10)$$

The entire charge on the sphere is thus given as:

$$q_1 = \int_0^{\pi}\int_0^{2\pi}\omega r_1^2 sin\theta d\theta d\psi \quad (11)$$

Now using the orthogonality property of Legendre functions, we find:

$$\int_0^{\pi}C_l^{\alpha/2-1}(cos\theta)C_{l'}^{\alpha/2-1}(cos\theta)sin\theta d\theta = 0, \quad when\ l \neq l' \quad (12)$$

where,

$C_0^{\alpha/2-1}(cos\theta) = 1\ and\ C_1^{\alpha/2-1}(cos\theta) = (\alpha - 2)cos\theta$

We take $l' = 0, C_0^{\alpha/2-1}(cos\theta) = 1$ and then $C_l^{\alpha/2-1}(cos\theta)$ vanishes when integrated for the limit 0 to π for all $l > 0$, thus we have:

$$q_1 = 4\pi\epsilon_2 r_1\phi_s - q(\alpha - 2)\frac{r_1}{\zeta} \quad (13)$$

Therefore, the potential of the sphere is given as:

$$\phi_s = \frac{1}{4\pi\epsilon_2}\frac{q_1}{r_1} + \frac{(\alpha-2)}{4\pi\epsilon_2}\frac{q}{\zeta} \quad (14)$$

Thus the potential for the integer order $\alpha = 3$ can be found as follows:

$$\phi_s = \frac{1}{4\pi\epsilon_2}\frac{q_1}{r_1} + \frac{1}{4\pi\epsilon_2}\frac{q}{\zeta} \quad (15)$$

where q_1 is showing the excess of the charge so it has been placed on the isolated sphere. Here, for a simple interpretation we choose the point $z = \zeta'$ such that $\zeta\zeta' = r_1^2$ is said to be the inverse of $z = \zeta$. In this way we can write,

$$\frac{1}{r_2'} = \sum_{l=0}^{\infty}\left(\frac{r^{2l}}{\zeta^l}\right)\frac{C_l^{\alpha/2-1}(cos\theta)}{r^{l+\alpha-2}}, (2 < \alpha \le 3) \quad (16)$$

Thus, the resultant potential is given as follows:

$$4\pi\epsilon_2\phi = \frac{q}{r_2} + \frac{q_1}{r} + \frac{qr_1}{\zeta}\frac{1}{r} - (\alpha-2)\frac{qr_1}{\zeta}\frac{1}{r_2'}, (2 < \alpha \le 3) \quad (17)$$

Next, we will determine electric potential due to point charge [16] outside and inside the sphere. At any point outside the sphere, the conductivity is zero and the inductive capacity is ϵ_1, thus the potential is given by:

$$\phi^+ = \phi_0 + \phi_1^+ = \frac{1}{4\pi\epsilon_2}\frac{q}{r_2} + \sum_{l=0}^{\infty}\frac{B_l}{r^{l+\alpha-2}}C_l^{\alpha/2-1}(\cos\theta), (2 < \alpha \le 3) \quad (18)$$

The symbol ϕ^+ shows the potential or field outside the sphere.

$$\phi^- = \sum_{l=0}^{\infty} A_l r^l C_l^{\alpha/2-1}(\cos\theta), \ (r < r_1) \quad (19)$$

The Notation ϕ^- represents the potential inside the sphere and the induced polarization.

$$\phi^+ = \sum_{l=0}^{\infty}\left[\frac{1}{4\pi\epsilon_2}\frac{q}{\zeta}\left(\frac{r}{\zeta}\right)^l + \frac{B_l}{r_1^{l+\alpha-2}}\right]C_l^{\frac{\alpha}{2}-1}(\cos\theta), (2 < \alpha \le 3) \quad (20)$$

Where,

$$\frac{1}{r_2} = \frac{1}{4\pi\epsilon_2}\frac{q}{\zeta}\sum_{l=0}^{\infty}\left(\frac{r}{\zeta}\right)^l C_l^{\alpha/2-1}(cos\theta)$$

Across the boundary, we find that:

$$\phi^+ = \phi^- \quad (21)$$

$$\epsilon_2\left[\frac{\partial\phi^+}{\partial r}\right]_{r=r_1} = \epsilon_1\left[\frac{\partial\phi^-}{\partial r}\right]_{r=r_1} \quad (22)$$

from the first boundary condition, we can calculate:

$$A_l r_1^l = \frac{1}{4\pi\epsilon_2}\frac{q}{\zeta^{l+1}}r_1^l + \frac{B_l}{r_1^{l+\alpha-2}} \quad (23)$$

2nd boundary condition leads us to the following:

$$\epsilon_1 A_l l r_1^{l-1} = \frac{1}{4\pi}\frac{q}{\zeta^{l+1}}l r_1^{l-1} - \epsilon_2(l+\alpha-2)\frac{B_l}{r_1^{l+\alpha-1}} \quad (24)$$

By simplification, we obtain the unknowns as follows:

$$B_l = \frac{qr_1^{2l+\alpha-2}}{4\pi\zeta^{l+1}}\frac{(\epsilon_2-\epsilon_1)}{\epsilon_2}\frac{l}{\epsilon_1 l+\epsilon_2(l+\alpha-2)} \quad (25)$$

and

$$A_l = \frac{q}{4\pi\zeta^{l+1}}\frac{2l+\alpha-2}{\epsilon_1 l+\epsilon_2(l+\alpha-2)} \quad (26)$$

At any point, the potential outside the sphere is given as:

$$\phi^+ = \frac{1}{4\pi\epsilon_2}\frac{q}{r_2} + \frac{q}{4\pi}\frac{(\epsilon_2-\epsilon_1)}{\epsilon_2}\sum_{l=0}^{\infty}\frac{l}{\epsilon_1 l+\epsilon_2(l+\alpha-2))}\frac{r_1^{2l+\alpha-2}}{\zeta^{l+1}}\frac{C_l^{\alpha/2-1}(\cos\theta)}{r^{l+\alpha-2}}, \ (2 < \alpha \le 3) \quad (27)$$

and also the potential inside the sphere is given as:

$$\phi^- = \frac{q}{4\pi\zeta^{l+1}}\sum_{l=0}^{\infty}\frac{2l+\alpha-2}{\epsilon_1 l+\epsilon_2(l+\alpha-2))}r^l C_l^{\frac{\alpha}{2}-1}(\cos\theta), \ (r < r_1) \quad (28)$$

For $l = 1$, at any point, the potential outside the sphere can be calculated as:

$$\phi^+ = \frac{1}{4\pi\epsilon_2}\frac{q}{r_2} + \frac{q}{4\pi}\frac{(\epsilon_2-\epsilon_1)}{\epsilon_2}\frac{1}{\epsilon_1+\epsilon_2(\alpha-1))}\frac{r_1^\alpha}{\zeta^2}\frac{(\alpha-2)\cos\theta}{r^{\alpha-1}}, \ (2 < \alpha \le 3) \quad (29)$$

and the potential inside the sphere is given as:

$$\phi^- = \frac{q}{4\pi\zeta^2}\frac{\alpha}{\epsilon_1+\epsilon_2(\alpha-1))}r(\alpha-2)\cos\theta, \ (2 < \alpha \le 3) \quad (30)$$

For $\alpha = 3$, we can find that the potential at any point outside the sphere is given as:

$$\phi^+ = \frac{1}{4\pi\epsilon_2}\frac{q}{r_2} + \frac{qr_1^3}{4\pi\zeta^2}\frac{(\epsilon_2-\epsilon_1)}{\epsilon_2(\epsilon_1+2\epsilon_2)}\frac{\cos\theta}{r^2} \quad (31)$$

and the potential inside the sphere is calculated as:

$$\phi^- = \frac{q}{4\pi\zeta^2}\frac{3}{\epsilon_1+2\epsilon_2}r\cos\theta \quad (32)$$

In this section [16], we find the electric potential and field for a dielectric sphere placed in FD Space. When the point source q deviates from the origin, the field in the vicinity of the sphere becomes parallel and homogeneous. Here, we consider a sphere placed in a host medium of electric constant ϵ_2 under the influence of a uniform parallel, and external field E_0, which is directed along the positive $z-axis$. Consequently, the primary potential can be written as follows:

$$\phi_0 = -E_0 z = -E_0 rcos\theta = -E_0 r P_1(cos\theta) \quad (33)$$

In FD space, the primary field is given:

$$\phi_0 = -E_0 z = -E_0 r C_1^{\frac{\alpha}{2}-1}(\cos\theta)$$

$$= -E_0 r(\alpha-2)\cos\theta \quad (34)$$

When ϕ_0 is not continuous at infinity, and for the source, it is itself infinitely remote. The potential at any point outside the sphere is because of either the induced surface charge or the polarization and is given as:

$$\phi_1^+ = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+\alpha-2}} C_l^{\alpha/2-1}(\cos\theta),\ (2 < \alpha \leq 3) \quad (35)$$

for conducting sphere, then the total potential on the surface of sphere as well as inside the sphere is a constant ϕ_s which is given as:

$$\phi_s = -E_0 r_1 C_1^{\alpha/2-1}(\cos\theta) + \sum_{l=0}^{\infty} \frac{B_l}{r_1^{l+\alpha-2}} C_l^{\alpha/2-1}(\cos\theta),\ (2 < \alpha \leq 3) \quad (36)$$

Since ϕ_s is independent of θ, so we can write:

$$B_0 = r_1\phi_s,\ B_1 = r_1^{\alpha} E_0,\ B_n = 0,\ for\ \ l > 1 \quad (37)$$

$$\phi^+ = -E_0 r(\alpha-2)\cos\theta + r_1^{\alpha} E_0 \frac{(\alpha-2)\cos\theta}{r^{\alpha-1}} + \frac{r_1\phi_s}{r} \quad (38)$$

Now we will compute the charge density and the total charge density is given as:

$$\omega = -\epsilon_2 \left(\frac{\partial\phi^+}{\partial r}\right)_{r=r_1} \quad (39)$$

Thus we can find the charge density as:

$$\omega = \alpha(\alpha-2)\epsilon_2 E_0 cos\theta + \frac{\phi_s r_1}{r}, q_1 = 4\pi\epsilon_2 r_1 \phi_s \quad (40)$$

When $\alpha = 3$,

$$\omega = 3\epsilon_2 E_0 \cos\theta + \frac{\phi_s r_1}{r},\ q_1 = 4\pi\epsilon_2 r_1\phi_s \quad (41)$$

The induced charge surface is due to a dipole moment $p = 4\pi\epsilon_2 r_1^{\alpha} E_0$.

In case. if the sphere is charged, then q_1 is added to the charge of the sphere. If the sphere is of a dielectric inductive capacity ϵ_1, then across the boundary conditions we can find the constants A_1 and B_1. As

$$\phi^+ = \phi^-,\ r = r_1 \quad (42)$$

$$\epsilon_2\left(\frac{\partial\phi^+}{\partial r}\right)_{r=r_1} = \epsilon_1\left(\frac{\partial\phi^-}{\partial r}\right)_{r=r_1} \quad (43)$$

Where $A_0 = 0$, $B_0 = 0$ and $B_n = 0$, $A_n = 0$, when $l > 1$. Then the boundary conditions yield:

$A_1 = \frac{-\alpha\epsilon_2 E_0}{\epsilon_1+(\alpha-1)\epsilon_2}$ and $B1 = \frac{\epsilon_1-\epsilon_2}{\epsilon_1+(\alpha-1)\epsilon_2} E_0 r_1^{\alpha}$

The resultant potentials are therefore given as:

$$\phi^+ = -E_0 r(\alpha-2)\cos\theta + \frac{\epsilon_1-\epsilon_2}{\epsilon_1+(\alpha-1)\epsilon_2} E_0 r_1^{\alpha} \frac{(\alpha-2)\cos\theta}{r^{\alpha-1}} \quad (44)$$

and

$$\phi^- = \frac{-\alpha\epsilon_2}{\epsilon_1+(\alpha-1)\epsilon_2} E_0(\alpha-2) r\cos\theta \quad (45)$$

where $z = r\cos\theta$, inside the sphere, the field $E^- = -\frac{\partial\phi}{\partial z}$ is paprallel and same. Thus,

$$E^- = \frac{\alpha\epsilon_2}{\epsilon_1+(\alpha-1)\epsilon_2} E_0(\alpha-2) \quad (46)$$

The dielectric constant κ_1 of the sphere is either greater or smaller than κ_2. Thus, the field within the spherical cavity is excited from a homogeneous dielectric constant κ_2 and it is given as:

$$E^- = \frac{\alpha\kappa_2}{1+(\alpha-1)\kappa_2} E_0(\alpha-2) > E_0 \quad (47)$$

Next, we see that the induced field outside the sphere is because of the dipole oriented along the direction of $z-axis$ whose dipole moment is given as:

$$p = 4\pi\epsilon_2 \frac{\epsilon_1-\epsilon_2}{\epsilon_1+(\alpha-1)\epsilon_2} E_0 r_1^{\alpha} \quad (48)$$

Apparently, the characteristics of a spherical cavity look like a dipole. This effect is readily accounted for revealing that the walls of the cavity bear a bound charge of density $\omega' = -n_1.P_2$, where P_2 is the polarization of the external medium. In the case of a dielectric sphere in the air, we have $\epsilon_2 = \epsilon_0$. The polarization of the sphere is then given as:

$$P_1 = \epsilon_0(\kappa_1-1)E^- = \alpha\frac{\kappa_1-1}{\kappa_1+(\alpha-1)}\epsilon_0 E_0 \quad (49)$$

and its dipole moment is then calculated as:

$$p = \frac{4}{3}\pi r_1^3.P_1 \quad (50)$$

The energy of the polarized sphere for the external field is thus given as:

$$U_1 = -\frac{1}{2}\int_v P_1 . E_0 dv = -\frac{1}{2} p . E_0 \quad (51)$$

3. CONCLUSION

In this article, the Laplace equation has been solved analytically for the fraction dimensional space. The non-integer dynamics plays a key role in describing the complex phenomenon. We have calculated an electrostatic potential of a conducting sphere as well as a dielectric sphere in fractional space. Moreover, we calculated an electric field and power radiated by the sphere. This is a general and close form solution that can be applied for various materials as host medium as well as core medium. We have checked that by setting alpha parameter equal to 3, the classical results can be recovered.

4. CONFLICT OF INTEREST

The authors declare no conflict of interest.

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