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Research Article

Fekete-Szegö Inequality and Radius Estimate for Certain Subclasses of Analytic Functions of Complex Order Associated with the Sine Function

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Abstract: In the present work, using suboridination techniques, we study certain subclasses of analytic univalent functions of complex order γ associated with the sine function. The sharp coefficient bounds and the sharp Fekete-Szegö inequality are obtained for these subclasses. Additionally, a sharp radius result is also derived for a certain subclass of Janowski starlike functions. These results generalize several known results as special cases and contribute to the broader theory of univalent functions.

Keywords: Analytic Functions, Coefficient Bounds, Fekete-Szegö Problem, Radius Problem, Sine Function, Subordination.

1. INTRODUCTION

In Geometric function theory (GFT) analytic functions and their geometric properties are regarded as core concepts. A key area of study within GFT is the class of univalent functions, which encompasses analytic functions. Many fields of mathematics and physics, including potential theory, fluid dynamics, and conformal mappings, rely heavily on these functions. where they are utilized to map the open unit disk onto domains with specific geometric structures [1]. Aleman and Constantin [2] proposed a significant connection between fluid dynamics and the theory of univalent functions. Specifically, the authors presented a straightforward technique that demonstrates how to explicitly solve the incompressible two-dimensional Euler equations using a univalent harmonic map. For the fundamental theory of univalent functions and a thorough understanding of the foundational concepts and key discoveries in the field, we direct the reader to previous studies [1, 3-5]. We will now review some fundamental definitions that are already familiar.

Let $\mathfrak{A}$ represent the family of analytic functions f defined in the open unit disk $\mathfrak{E} = \{\psi \in \mathbb{C}: |\psi| < 1\}$ that satisfying $f(0) = 0$, $f'(0) = 1$ and admit the following power series expansion:

$$f(\psi) = \psi + \sum_{n=2}^{\infty} a_n \psi^n. \tag{1}$$

A function $f \in \mathfrak{A}$ is said to be univalent in $\mathfrak{E}$ if it maps distinct points in $\mathfrak{E}$ to distinct points in its image, i.e., $f(\psi_1) = f(\psi_2)$ implies $\psi_1 = \psi_2$ for $\psi_1, \psi_2 \in \mathfrak{E}$. The class of all such univalent functions is denoted by $\mathcal{S}$. It is evident that an analytical function u for which $u(0) = 0$ and $|u(\psi)| < 1$ is called Schwartz function. Let g_1

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and g_2 be analytic in $\mathfrak{E}$. Then g_1 is subordinate to g_2, written as $g_1 \prec g_2$, if there exists a Schwartz function u, such that $g_1(\psi) = g_2(u(\psi))$.

Several important subclasses of univalent functions have been introduced and studied, from which the classes S^* and C, consisting of starlike and convex univalent functions respectively, are considered as the most extensively investigated subclasses of the class $\mathcal{S}$ of univalent functions in the open unit disk $\mathfrak{E} = \{\psi \in \mathbb{C}: |\psi| < 1\}$, and have been of major interest. Ma and Minda [6] introduced a class of analytic and univalent functions $\varphi(\psi)$ and proposed the following subclasses of analytic functions.

$$S^*(\varphi) = \left\{f \in \mathfrak{A}: \frac{\psi f'(\psi)}{f(\psi)} \prec \varphi(\psi)\right\}, \tag{2}$$

and

$$C(\varphi) = \left\{f \in: \mathfrak{A}\ 1 + \frac{\psi f''(\psi)}{f'(\psi)} \prec \varphi(\psi)\right\}. \tag{3}$$

Where $\varphi(\psi)$ is a univalent function with $\varphi(0) = 1$, $\varphi'(0) > 1$ and the region $\varphi(\mathfrak{E})$ is symmetric with respect to the real axis and star-shaped around the point $\varphi(0) = 1$.

An extension of equations (2) and (3), respectively, was provided by Ravichandran *et al.* [7], as follows:

$$S^*(\gamma, \varphi) = \left\{f \in \mathfrak{A}: 1 + \frac{1}{\gamma}\left(\frac{\psi f'(\psi)}{f(\psi)} - 1\right) \prec \varphi(\psi)\right\},$$

and

$$C(\gamma, \varphi) = \left\{f \in \mathfrak{A}: 1 + \frac{1}{\gamma}\left(\frac{\psi f''(\psi)}{f'(\psi)}\right) \prec \varphi(\psi)\right\}.$$

where $\gamma \in \mathbb{C} \setminus \{0\}$. Such functions are commonly referred to as Ma–Minda type functions of order γ , $(\gamma \in \mathbb{C} \setminus \{0\})$.

A central focus in this area is the investigation of the coefficient bounds and related inequalities for various subclasses of $\mathcal{S}$, including results involving the Fekete-Szegö functional $|a_3 - \lambda a_2^2|$, for $0 < \lambda < 1$ [8], and the radius problems. We now provide a brief review of these investigations.

Finding bounds for function coefficients is a central problem in GFT, as it governs growth and distortion properties. The second coefficient, for instance, controls growth and distortion. Reformulated as the task of estimating a_n, the coefficient problem has been among the most challenging in the field. In 1916, Bieberbach [9] conjectured that for $f \in \mathcal{S}$and has the form, given in equation (1), then then $|a_n| \leq n,\ (n \geq 2)$. The conjecture was finally settled in 1985 by Louis de Branges [10] confirming it for all $n \geq 2$. The Fekete-Szegö problem is a classical coefficient problem that seeks sharp bounds for linear combinations of coefficients, typically $|a_3 - \lambda a_2^2|$, offering detailed information on the geometric characteristics of subclasses of univalent functions. Many studies have since focused on these problems; some recent contributions can be seen in previous studies [11-15].

Finally, attention is given to the radius problems, where the aim is to determine the maximal radius for which functions belonging to a particular subclass of $\mathcal{S}$ preserve specific geometric properties such as starlikeness or convexity. If we consider certain transformations or geometric conditions that fail to maintain univalence, such as those within the unit disk. It naturally leads to the question of whether such transformations or conditions may uphold univalence in a smaller subdisk

$\mathfrak{E}_0 = \{\psi: |\psi| < R < 1\} \subset \mathfrak{E}$.

Problems of this nature are commonly referred to as "radius problems". The radius, R, represents the largest subdisk, $\mathfrak{E}_0$ within which specific transformations of a univalent function f or certain geometric conditions ensure univalence. The radius problems ensure the validity of certain geometric properties (such as starlikeness or convexity) within a disk of maximal size [16].

A key tool in such investigations is the well-known Carathéodory class $\mathcal{P}$, consisting of analytic functions p in $\mathfrak{E}$, and has the following form:

$$p(\psi) = 1 + \sum_{n=1}^{\infty} c_n \psi^n;\ (\psi \in \mathfrak{E}). \tag{4}$$

With $Re(p(\psi)) > 0$. A significant portion on coefficient problems involves expressing coefficients of functions in a given class in terms of those with positive real part, using known inequalities for $\mathcal{P}$ to analyze them. To unify and extend the study of these and related subclasses, Ma–Minda [6] form of analytic functions are used, which offers a unified approach to several subclasses of analytic functions through subordination techniques.

By following unified approach presented by Ma and Minda [6], several authors in recent years have proposed new subclasses of the class $\mathcal{S}$ of univalent functions by selecting specific forms of the analytic function $\varphi(\psi)$. Examples of such choices for $\varphi(\psi)$ include: $S^*[A_1, A_2] = S^*\left(\frac{1+A_1\psi}{1+A_2\psi}\right)$, $-1 \leq A_2 < A_1 \leq 1$, is a widely studied the Janowski subclass of starlike functions [17], and the class of starlike function of order $\propto$ deonted by $S^*(\propto) = S^*\left(\frac{1+(1-2\propto)\psi}{1-\psi}\right), 0 \leq \propto < 1$. Similarly the class of Janowski convex functions and the class of convex function of order $\propto$,respectively expressed as $C[A_1, A_2] = C\left(\frac{1+A_1\psi}{1+A_2\psi}\right)$ and $C(\propto) = C\left(\frac{1+(1-2\propto)\psi}{1-\psi}\right)$ In particular, $S^*(0) = S^*$ and $C(0) = C$ are the classical classes of starlike and convex functions, respectively. Cho *et al*. [18] introduced the class S^*_{sin} of analytic functions, defined as follows:

$$S^*_{sin} = \left\{ f \in \mathfrak{A}\colon \frac{\psi f'(\psi)}{f(\psi)} \prec 1 + sin\psi;\ (\psi \in \mathfrak{E}) \right\}.$$

In their work, authors conducted a comprehensive study on various radius problems associated with the class S^*_{sin} of analytic functions. Their analysis includes determining sharp bounds for radii associated with geometric properties such as starlikeness and convexity. Further contributions to this class can be found in the work of Arif *et al.* [19], where the authors obtained sharp coefficient bounds, investigated the Fekete-Szegö inequality, and examined the Hankel determinant of order three. Additional geometric properties of functions in the class S^*_{sin}, along with recent related developments, are discussed by Srivastava *et al*. [20] and Tang *et al*. [21]. Recently, Al-Shaikh *et al.* [22] introduced a class of starlike functions of complex order γ, denoted by $\mathcal{L}(m, n, \propto, \gamma)$ where $-1 < n < m \leq 1$, $0 \leq \propto \leq 1$, and $\gamma \in \mathbb{C} \setminus \{0\}$. This class comprises functions $f \in \mathfrak{A}$ that satisfy the subordination condition as follows:

$$1 + \frac{1}{\gamma}\left(\frac{\psi f'(\psi) + \propto \psi^2 f''(\psi)}{(1-\propto) f(\psi) + \propto \psi f'(\psi)} - 1\right) \prec \varphi_{Car}(m, n; \psi),$$

where $\varphi_{Car}(m, n; \psi)$ defines a cardioid domain, given by:

$$\varphi_{Car}(m, n; \psi) = \frac{2m\tau^2\psi^2 + (m-1)\tau\psi + 2}{2n\tau^2\psi^2 + (n-1)\tau\psi + 2},$$

with $\tau = \frac{1-\sqrt{5}}{2}$, and $\psi \in \mathfrak{E}$.

2. PROPOSED WORK

Based on the earlier works of Cho *et al.* [18], Arif *et al.* [19], and Al-Shaikh *et al.* [22], the present investigation introduces a new subclass $\mathfrak{R}_{sin}(\propto, \gamma; \varphi_{sin})$ of analytic functions of complex order γ.

Definition 1. If a function $f \in \mathfrak{A}$ has the form given in (1), then it belongs to the subclass $\mathfrak{R}_{sin}(\propto, \gamma; \varphi_{sin})$, provided the following conditions are satisfied:

$$1 + \frac{1}{\gamma}\left\{\frac{\psi f'(\psi) + \propto \psi^2 f''(\psi)}{(1-\propto) f(\psi) + \propto \psi f'(\psi)} - 1\right\} \prec \varphi_{sin}(\psi),$$

for $0 \leq \propto \leq 1, \gamma \in \mathbb{C} \setminus \{0\}$ and $\varphi_{sin}(\psi) = 1 + sin\psi;\ (\psi \in \mathfrak{E})$.

Equivalently, $f \in \mathfrak{R}_{sin}(\propto, \gamma; \varphi_{sin})$ if the image of

$$1 + \frac{1}{\gamma}\left\{\frac{\psi f'(\psi) + \propto \psi^2 f''(\psi)}{(1-\propto) f(\psi) + \propto \psi f'(\psi)} - 1\right\},$$

lies within the eight-shaped region φ_{sin} located in the right half-plane (see Figure 1 reported by Cho *et al.* [18]). Several known and newly defined subclasses emerge as special cases of Definition 1 by choosing particular values of the parameters $\propto$ and γ, as illustrated below:

(i) $\mathfrak{R}_{sin}(0, \gamma; \varphi_{sin}) \equiv \mathfrak{S}^*_{sin}(\gamma; \psi)$: A new subclass of starlike functions characterized by a complex parameter γ, related to the sine function.
(ii) $\mathfrak{R}_{sin}(0,1; \varphi_{sin}) \equiv S^*_{sin}$: A subclass of starlike functions associated with the sine function introduced by Cho *et al.* [18].
(iii) $\mathfrak{R}_{sin}(1, \gamma; \varphi_{sin}) \equiv \mathfrak{C}_{sin}(\gamma; \psi)$: A new subclass of convex functions of complex γ linked with the sine function.
(iv) $\mathfrak{R}_{sin}(1,1; \varphi_{sin}) \equiv C_{sin}$: A subclass of convex functions associated with the sine function introduced by Arif *et al.* [19].

3. SET OF LEMMAS

We begin by stating the following auxiliary results, that will be employed in deriving the main results.

Lemma 1. If $p \in \mathcal{P}$ and has the form given in (1), then

$$|c_n| \leq 2\ for\ n \geq 1, \tag{5}$$

and for any $v \in \mathbb{C}$,

$$|c_2 - vc_1^2| \leq 2\,max\{1,\ |2v - 1|\}. \tag{6}$$

The relation in equation (5) can be found in previously reported literature [23]. The inequality stated in equation (6) was established by Keogh and Merkes [24] and related discussion is given by Ma and Minda [6].

Lemma 2. [25] Let, $p \in \mathcal{P}(\propto)$. Then for $|\psi| = r$, we have

$$\left|\frac{\psi p'(\psi)}{p(\psi)}\right| \le \frac{2(1-\propto)nr^n}{(1-r^n)(1+(1-2\propto)r^n)}.$$

Lemma 3. [18] Let $1 - sin1 \le a \le 1 + sin1$ and $r_a = sin1 - |a-1|$. The following hold.

$\{\mathcal{W} \in \mathbb{C}: |\mathcal{W} - a| < r_a\} \subset \delta_{sin} \subset \{\mathcal{W} \in \mathbb{C}: |\mathcal{W} - 1| < sin1\}$.

Where $\delta_{sin} = \varphi_{sin}(\mathfrak{E})$.

4. MAIN RESULTS

In this section, we investigate the sharp bounds for the first two coefficients and obtain sharp estimates for the Fekete-Szegö inequality for functions belonging to the class $\mathfrak{R}_{sin}(\propto, \gamma; \varphi_{sin})$. Additionally, a sharp $\mathfrak{S}^*_{sin}(\gamma;\psi)$ −radius result is also derived for a certain subclass of Janowski starlike functions. Known subclasses and their corresponding results are also discussed to emphasize the connection between the present work and previous studies.

4.1. Coefficint Problems

Theorem 1. If the function $f \in \mathfrak{R}_{sin}(\propto, \gamma; \varphi_{sin})$, and be given by equation (1). Then,

$$|a_2| \le \frac{|\gamma|}{(1+\propto)},$$

$$|a_3| \le \frac{|\gamma|}{2(1+2\propto)}(1 + |\gamma - 1|).$$

These findings are sharp.

Proof. If $f \in \mathfrak{R}_{sin}(\propto, \gamma; \varphi_{sin})$, then by defintion we have:

$$1 + \frac{1}{\gamma}\left\{\frac{\psi f'(\psi) + \propto \psi^2 f''(\psi)}{(1-\propto)f(\psi) + \propto \psi f'(\psi)} - 1\right\} = 1 + sin(u(\psi)), \quad (7)$$

where $u(\psi)$ is a Schwarz function. Let, p be a function such that:

$$p(\psi) = \frac{1 + u(\psi)}{1 - u(\psi)} = 1 + c_1\psi + c_2\psi^2 + c_3\psi^3 + \cdots, \quad (8)$$

then $p(\psi) \in \mathcal{P}$. It follows that:

$$u(\psi) = \frac{p(\psi) - 1}{p(\psi) + 1} = \frac{c_1\psi + c_2\psi^2 + c_3\psi^3 + \cdots}{2 + c_1\psi + c_2\psi^2 + c_3\psi^3 + \cdots}.$$

Using equation (7), we obtain:

$$1 + \frac{1}{\gamma}\left\{\frac{\psi f'(\psi) + \propto \psi^2 f''(\psi)}{(1-\propto)f(\psi) + \propto \psi f'(\psi)} - 1\right\}$$

$$= 1 + \frac{(1+\propto)a_2}{\gamma}\psi + \frac{(2(1+2\propto)a_3 - (1+\propto)^2 a_2^2)}{\gamma}\psi^2$$

$$+ \frac{[3(1+\propto)a_4 - 3(1+\propto)(1+2\propto)a_2a_3 + (1+\propto)^3 a_2^3]}{\gamma}\psi^3 + \cdots \quad (9)$$

By expanding $u(\psi)$ into its series form and applying elementary calculations, we obtain:

$$1 + sin(u(\psi)) = 1 - u(\psi) + \frac{(u(\psi))^3}{3!} - \frac{(u(\psi))^5}{5!} + \cdots$$

$$= 1 + \frac{1}{2}c_1\psi + \left(\frac{c_2}{2} - \frac{c_1^2}{4}\right)\psi^2 + \left(\frac{5c_1^3}{48} + \frac{c_3}{2} - \frac{c_1c_2}{2}\right)\psi^3 + \cdots \quad (10)$$

From equations (9), and (10), we obtain

$$a_2 = \frac{\gamma}{(1+\propto)}c_1, \quad (11)$$

and

$$a_3 = \frac{\gamma}{4(1+2\propto)}c_2 + \frac{\gamma}{4(1+2\propto)}c_1^2(\gamma - 1). \quad (12)$$

Applying the relation (5) in (11), we obtain the required result. Furthermore, rearranging equation (12) gives:

$$|a_3| = \frac{1}{4(1+2\propto)}\left|\gamma c_2 + \gamma c_1^2(\gamma - 1)\right|.$$

Applying the triangle inequality together with equation (5) leads to the desired result. Sharpness of the bounds can be demonstrated by the following function, corresponding to particular values of the parameters $\propto$ and γ.

$$f(\psi) = \psi\, exp\left(\int_0^{\psi} \frac{sint - 1}{t} dt\right)$$

$$= \psi + \psi^2 + \frac{\psi^3}{2} + \frac{\psi^4}{9} + \cdots. \quad (13)$$

Remark 1. For $\propto = 0$ in Theorem 1, we obtain the following new result for the subclass $\mathfrak{S}^*_{sin}(\gamma;\psi)$.

Theorem 2. Let $f \in \mathfrak{S}^*_{sin}(\gamma;\psi)$ and be given by (1), then
$|a_2| \leq |\gamma|$, and $|a_3| \leq \frac{|\gamma|}{2}(1+|\gamma-1|)$.
These bounds are sharp.

Remark 2. For $\propto = 0$ and $\gamma = 1$ in Theorem 1, we obtain the following known result for the subclass S^*_{sin}, investigted by Cho *et al.* [18].

Theorem 3. Let $f \in S^*_{sin}$, be given by equation (1) Then,

$|a_2| \leq 1$, and $|a_3| \leq \frac{1}{2}$.
Equality in these bounds is attained by the function (13).

Remark 3. Setting $\propto = 1$ in Theorem 1 yields a new result for the subclass $\mathfrak{C}_{sin}(\gamma;\psi)$, as stated below:

Theorem 4. Let $f \in \mathfrak{C}_{sin}(\gamma;\psi)$ be given by (1) Then,
$|a_2| \leq \frac{|\gamma|}{2}$, and $|a_3| \leq \frac{|\gamma|}{6}(1+|\gamma-1|)$.
The bounds are sharp for $\gamma = 1$ and equality is attained by the function:

$$1+\frac{1}{\gamma}\left\{\frac{\psi f''(\psi)}{f'(\psi)}\right\} = 1+sin(\psi). \tag{14}$$

Remark 4. For $\propto = 1$ and $\gamma = 1$ in Theorem 1, we obtain the following known result for the subclass C_{sin}, studied by Arif *et al.* [19].

Theorem 5. Let $f \in C_{sin}$, and be given by equation (1). Then,

$|a_2| \leq \frac{1}{2}$, and $|a_3| \leq \frac{1}{6}$.

Equality in these bounds are attained by the function described in equation (14).

4.2. Fekete-Szegö Problem

We now investigate the Fekete-Szegö functional for functions belonging to the subclass $\mathfrak{R}_{sin}(\propto,\gamma;\varphi_{sin})$

Theorem 6. If a function f of the form equation (1) belongs to $\mathfrak{R}_{sin}(\propto,\gamma;\varphi_{sin})$ then
$$|a_3-\lambda a_2^2| \leq \frac{|\gamma|}{2(1+2\propto)}\, max\left\{1, \left|\frac{2\lambda(1+2\propto)-(2\gamma-1)(1+\propto)^2}{(1+\propto)^2}\right|\right\}.$$
This result is sharp.

Proof. If $f \in \mathfrak{R}_{sin}(\propto,\gamma;\varphi_{sin})$, then from equations (11) and (12), we obtain: $a_3-\lambda a_2^2$
$$= \frac{\gamma}{4(1+2\propto)}\left\{c_2-c_1^2\left(\frac{\lambda(1+2\propto)-(\gamma-1)(1+\propto)^2}{(1+\propto)^2}\right)\right\}.$$

Let,
$$v = \left(\frac{\lambda(1+2\propto)-(\gamma-1)(1+\propto)^2}{(1+\propto)^2}\right).$$

Therefore:
$$|a_3-\lambda a_2^2| = \frac{|\gamma|}{4(1+2\propto)}\left|c_2-vc_1^2\right|,$$

from Lemma 1, equation (6), it follows that for v, we obtained the required result. for specific values of $\propto$ and γ, equality is achieved by the function defined in equation (13).

Remark 5. A new result for the subclass $\mathfrak{S}^*_{sin}(\gamma;\psi)$ is obtained by setting $\propto = 0$ in Theorem 6, as follow:

Theorem 7. Let $f \in \mathfrak{S}^*_{sin}(\gamma;\psi)$ be given (1). Then
$$|a_3-\lambda a_2^2| \leq \frac{|\gamma|}{2}\, max\{1, |2\lambda-(2\gamma-1)|\}.$$

Equality is achieved for $\gamma = 1$ by the function given in equation (13).

Remark 6. The choice $\propto = 0$ and $\gamma = 1$ in Theorem 6 gives the result established earlier for the class S^*_{sin} by Arif *et al.* [19].

Theorem 8. Suppose $f \in S^*_{sin}$ is be given (1). Then
$$|a_3-\lambda a_2^2| \leq \frac{1}{2}\, max\{1, |2\lambda-1|\}.$$
The result attains the best possible bound.

Remark 7. A new result and its corresponding corollary for the subclass $\mathfrak{C}_{sin}(\gamma;\psi)$ are obtained by setting $\propto = 1$ and $\lambda = 1$ in Theorem 6, respectively, as follows:

Theorem 9. Let $f \in \mathfrak{C}_{sin}(\gamma;\psi)$ be given (1). Then
$$|a_3-\lambda a_2^2| \leq \frac{|\gamma|}{6}\, max\left\{1, \frac{1}{2}|3\lambda-2(2\gamma-1)|\right\}.$$

Corollary 1. If $f \in \mathfrak{C}_{sin}(\gamma;\psi)$ with series form (1),
$$|a_3-a_2^2| \leq \frac{|\gamma|}{6}.$$

The inequality becomes sharp for the specific case $\gamma = 1$.

Remark 8. By setting $\propto = 1$, $\gamma = 1$ and $\lambda = 1$ in Theorem 6, we obtain the following known result and corollary for the class C_{sin}, investigated Arif *et al*. [19].

Theorem 10. Let $f \in C_{sin}$ be given (1). Then

$$\left|a_3 - \lambda a_2^2\right| \leq \frac{1}{6}\, max\left\{1, \frac{1}{2}|3\lambda - 2|\right\}.$$

Corollary 2. If $f \in C_{sin}$ with series form (1), then $\left|a_3 - a_2^2\right| \leq \frac{1}{6}$.
This inequality is sharp.

4.3. Radius Problem

In order discuss about the findings in this section, we first highlight a few known classes as follows: Let for $-1 \leq A_2 < A_1 \leq 1$,

$$\mathcal{P}[A_1, A_2] = \left\{p(\psi) = 1 + \sum_{k=n}^{\infty} c_n\, \psi^n : p(\psi) \prec \frac{1 + A_1\psi}{1 + A_2\psi}\right\}$$

denoted by

$$\mathcal{P}(\propto) = \mathcal{P}[1 - 2\propto, -1]\ and\ \mathcal{P} = \mathcal{P}(0).$$

For $f \in \mathfrak{A}$, if $p(\psi) = \frac{\psi f(\psi)}{f'(\psi)}$, then $\mathcal{P}[A_1, A_2]$ is represented by $S^*[A_1, A_2]$ and $S^*(\propto) = S^*[1 - 2\propto, -1]$. For this class, Cho *et al*. [18], determined the S_{sin}^* −radii for the class of Janowski starlike functions and some other geometrically defined classes including the classes

$\mathfrak{S}_n = \left\{f \in \mathfrak{A} : \frac{f(\psi)}{\psi} \in \mathcal{P}\right\}$, $S^*[A_1, A_2]$ and

$$\mathcal{C}\mathfrak{S}_n = \left\{f \in \mathfrak{A}: \frac{f(\psi)}{g(\psi)} \in \mathcal{P},\ g \in S^*(\propto)\right\}$$

these classes were considered by Ali *et al.* [26].

This section aims to determine the $\mathfrak{S}_{sin}^*(\gamma; \varphi_{sin})$ −radius result for the subclass $\mathfrak{S}_n$ of Janowski starlike functions .

Theorem 11. The sharp $\mathfrak{S}_{sin}^*(\gamma; \varphi_{sin})$ −radius for the subclass $\mathfrak{S}_n$ is defined as:

$$R_{\mathfrak{S}_{sin}^*(\gamma;\varphi_{sin})}(\mathfrak{S}_n) = \left(\frac{sin1}{\sqrt{|\gamma|^2 n^2 + sin^2(1)} + |\gamma|n}\right).$$

Proof. Let $f \in \mathfrak{S}_n$, define $h(\psi) = \frac{f(\psi)}{\psi}$, so that $h \in \mathcal{P}$. Then

$$\frac{\psi f'(\psi)}{f(\psi)} - 1 = \frac{\psi h'(\psi)}{h(\psi)}.$$

Now, from the definition of the class $\mathfrak{S}_{sin}^*(\gamma; \varphi_{sin})$, we consider the function:

$$\chi(\psi) = 1 + \frac{1}{\gamma}\left(\frac{\psi f'(\psi)}{f(\psi)} - 1\right) = 1 + \frac{1}{\gamma}\left(\frac{\psi h'(\psi)}{h(\psi)}\right).$$

From Lemma 2, since $h \in \mathcal{P}$, it follows that:

$$\left|\frac{\psi h'(\psi)}{h(\psi)}\right| \leq \frac{2nr^n}{1 - r^{2n}}\ for\ |\psi| = r.$$

Thus,

$$|\chi(\psi) - 1| \leq \frac{2nr^n}{|\gamma|(1 - r^{2n})}.$$

From Lemma 3, the image of the function $\varphi_{sin}(\psi) = 1 + sin\psi$ contains the open disk:

$$\delta_{sin} = \{\mathcal{W} \in \mathbb{C}: |\mathcal{W} - 1| < sin1\}.$$

To ensure that $\chi(\psi) \prec 1 + sin\psi$, we require:

$$\frac{2nr^n}{|\gamma|(1 - r^{2n})} \leq sin1,$$

or equivalently, $|\gamma|(sin1)r^{2n} + 2nr^n - |\gamma|(sin1) \leq 0$.
Thus, the sharp radius R corresponds to the smallest positive root of the equation:

$$|\gamma|(sin1)r^{2n} + 2nr^n - |\gamma|(sin1) = 0.$$

To show sharpness, consider the function:

$$f_o(\psi) = \frac{1 + \psi^n}{1 - \psi^n} \in \mathcal{P},$$

then

$$\frac{\psi f_o'(\psi)}{f_o(\psi)} = 1 + \frac{2n\psi^n}{\gamma(1 - \psi^{2n})}$$

$$\Rightarrow |f_o(\psi) - 1| = \frac{2n\psi^n}{|\gamma|(1 - \psi^{2n})} = sin1.$$

This completes the proof.

Remark 9. When $\gamma = 1$, Theorem 11 coincides with the known sharp radius result for the subclass

S^*_{sin}, examineded by Cho *et al.* [18].

Theorem 12. The sharp S^*_{sin} −radius for the subclass $\mathfrak{S}_n$ is given by:

$$R_{S^*_{sin}}(\mathfrak{S}_n) = \left(\frac{sin1}{\sqrt{n^2 + sin^2(1) + n}}\right).$$

5. CONCLUSIONS

In this study, we introduce some new subclasses of analytic functions of complex order $\boldsymbol{\gamma}$, defined via subordination involving the sine function. Sharp coefficient estimates, Fekete-Szegö inequalities, and a radius result for a certain class are also obtained. These findings not only support applications in conformal mapping and applied analysis but also enhance the current theory of univalent functions. Furthermore, a well-known class and its associated results are highlighted to demonstrate the connection between earlier research and the present findings.

6. CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

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