Proceedings of the Pakistan Academy of Sciences: A Physical and Computational Sciences 61(3): 273-281 (2024)

ISSN (Print): 2518-4245; ISSN (Online): 2518-4253
http://doi.org/10.53560/PPASA(61-3)852

Pakistan Academy of Sciences

Research Article

An Efficient Four Step Fifteenth Order Method for Solution of Non-Linear Models in Real-World Problems

Sanaullah Jamali[1]*, Zubair Ahmed Kalhoro[1], and Imran Qasim Memon[2]

[1]Institute of Mathematics and Computer Science, University of Sindh, Allama I.I. Kazi Campus, Jamshoro, Sindh, Pakistan

[2]Department of Basic Sciences and Related Studies, Mehran University of Engineering and Technology, Jamshoro, Sindh, Pakistan

Abstract: Non-linear equations are fundamental to a wide range of practical applications in engineering and applied sciences. This research paper presents a novel iterative scheme—a fifteenth-order approach—designed to effectively solve non-linear problems. The numerical results of the Proposed Scheme are thoroughly compared with those of existing methods. Graphical representations and basin of attraction analysis reveal that the fifteenth-order method achieves superior accuracy and efficiency, surpassing alternative methods in the precise estimation of solutions to non-linear problems.

Keywords: Real Word Problems, Taylor Series Expansion, Order of Convergence, Fifteenth Order Methods, Basin of Attraction.

1. INTRODUCTION

The prevalence of non-linear equations, as represented by the form $\emptyset(\varkappa) = 0$, is observed across various fields in both pure and applied sciences, engineering, and computing. Recent efforts have been dedicated by numerous scientists and engineers to the addressing of non-linear equations through both numerical and analytical methods [1]. In the field of literature, various Iterative Schemes and algorithms have been developed, drawing inspiration from techniques such as interpolation, Taylor's series, quadrature formulas, and decomposition. Additionally, there are several modifications and enhancements to existing methods, along with the introduction of various hybrid iterative approaches. Xiaojian [2] introduced several variants of the Chebyshev–Halley methods that do not require the second derivative. The proposed methods were demonstrated to have a minimum convergence order of three, with each iteration involving the evaluation of two functions and one first derivative. Lakho *et al.* [3] presented a seventh ordered three step method using Lagrange interpolation technique. Naseem *et al.* [4] proposed fourth method based on forward and finite difference schemes. Abbasi *et al.* [5] proposed a ninth order method using Hermite interpolation technique. Soomro *et al.* [6] proposed a bracketing approach based on the Regula Falsi method, leading to an enhanced convergence rate. Generally, the expression of the roots of non-linear or transcendental equations in closed form or their analytical computation is found to be challenging. The calculation of approximations of the roots, typically presented as either small isolating intervals or floating-point numbers, is enabled by root-finding algorithms.

In this paper, a hybrid four-step scheme for the solution of non-linear methods is presented. In this scheme, the convergence rate is enhanced, and the number of function evaluations required per iteration is reduced through the incorporation of a weight function and interpolation techniques, respectively, contributing to the improvement of

Received: April 2024; Revised: August 2024; Accepted: September 2024
* Corresponding Author: Sanaullah Jamali <sanaullah.jamali@usindh.edu.pk>

the efficiency index. To evaluate the effectiveness of the proposed approach, numerous examples and Real-world problems were subjected to testing and comparison with existing methods (TE1 15th, TE2 15th, and TE3 15th as mentioned by Eftekhari [7], and FS 15th as indicated by Soleymani and Sharifi [8]). The results of these comparisons are presented in various Tables and Figures, while the stability of the Proposed Scheme is further validated through the analysis of the Basin of attraction. This method is not applicable for the system of nonlinear equations.

2. DERIVATION OF PROPOSED SCHEME

For the first step of Proposed Scheme we take Newton method [9, 10].

$$\nu_i = \varkappa_i - \frac{\phi(\varkappa_i)}{\phi'(\varkappa_i)} \tag{1}$$

And for the second step of Proposed Scheme we take again newton method Kumar *et al.* [11], reduce the function evaluations we modify it by replacing the $\phi'(\nu_i)$ by $\phi'(\varkappa_i)$ in second step and introduce weight function P

$$\left.\text{And } P = 1 + 2a + a^2 + a^3, \quad a = \frac{\phi(\nu_i)}{\phi(\varkappa_i)}\right\}$$

$$\xi_i = \nu_i - P\left(\frac{\phi(\nu_i)}{\phi'(\varkappa_i)}\right) \tag{2}$$

For the third step of Proposed Scheme we take again newton method [11], to reduce the function evaluations we replacing the $\phi'(\xi_i)$ by $H'(\xi_i)$ Hermite interpolation in third step.

$$o_i = \xi_i - \frac{\phi(\xi_i)}{H'(\xi_i)}, \tag{3}$$

In three-point formula (3) it requires five function evaluation per iteration, to reduce the number of function evaluations we approximate $\phi'(\xi_i)$ using available data. Since we have four values $\phi(\varkappa)$, $\phi'(\varkappa), \phi(\nu)$ & $\phi(\xi)$ approximate ϕ by its Hermite's interpolating polynomial $H_3$of degree 3 at the nodes $\varkappa, \nu, \xi$ and utilize the approximation $\phi'(\xi) \approx H_3'(\xi)$ in the third step of the iterative scheme (3). Hermite's interpolating polynomial of third degree has the form.

$$H_3(\eta) = a_0 + a_1(\eta - \varkappa) + a_2(\eta - \varkappa)^2 + a_3(\eta - \varkappa)^3 \tag{4}$$

And its derivative is:

$$H_3'(\eta) = a_1 + 2a_2(\eta - \varkappa) + 3a_3(\eta - \varkappa)^2 \tag{5}$$

The unknown coefficients will be determined using available data from the conditions:

$$H_3(\varkappa) = \phi(\varkappa), H_3(\nu) = \phi(\nu), H_3(\xi) = \phi(\xi), \& \ H_3'(\varkappa) = \phi'(\varkappa)$$

Putting $\eta = \varkappa$ into (4) & (5) we get $a_0 = \phi(\varkappa)$ and $a_1 = \phi'(\varkappa)$. The coefficients a_2 and a_3 are obtained from the system of two linear equation formed by using the remaining two conditions $\eta = \nu$ & $\eta = \xi$ in (4) and we obtained.

$$a_2 = \frac{(\xi-\varkappa)\phi[\nu,\varkappa]}{(\xi-\nu)(\nu-\varkappa)} - \frac{(\nu-\varkappa)\phi[\xi,\varkappa]}{(\xi-\nu)(\nu-\varkappa)} - \phi'(\varkappa)\left(\frac{1}{\xi-\varkappa} - \frac{1}{\nu-\varkappa}\right)$$

$$\& \quad a_3 = \frac{\phi[\xi,\varkappa]}{(\xi-\nu)(\xi-\varkappa)} - \frac{\phi[\nu,\varkappa]}{(\xi-\nu)(\nu-\varkappa)} + \frac{\phi'(\varkappa)}{(\xi-\varkappa)(\nu-\varkappa)}$$

By putting the values of a_1, a_2, a_3 & $\eta = \xi$ in (5) we get:

$$H_3'(\xi) = 2(\phi[\varkappa,\xi] - \phi[\varkappa,\nu]) + \phi[\nu,\xi] + \frac{\nu-\xi}{\nu-\varkappa}(\phi[\varkappa,\nu] - \phi'(\varkappa)) \tag{6}$$

Finally, in fourth step of Proposed Scheme we take again newton method from Kumar *et al.* [11], to reduce the function evaluations we replacing the $\phi'(o_i)$ by $j'(o_i)$ in four step.

$$\varkappa_{i+1} = o_i - \frac{\phi(o_i)}{j'(o_i)} \tag{7}$$

where $j'(o_i) = \phi[\varkappa_i,\xi_i] + (\phi[\nu_i,\varkappa_i,\xi_i] - \phi[\nu_i,\varkappa_i,o_i] - \phi[\xi_i,\varkappa_i,o_i])(\nu_i - \xi_i)$

see in [7].

$$\left.\begin{aligned} &\text{Step. 1 } \nu_i = \varkappa_i - \frac{\phi(\varkappa_i)}{\phi'(\varkappa_i)} \\ &\text{Step. 2 } \xi_i = \nu_i - P\left(\frac{\phi(\nu_i)}{\phi'(\varkappa_i)}\right) \\ &\text{Step. 3 } o_i = \xi_i - \frac{\phi(\xi_i)}{H_3'(\xi_i)} \\ &\text{Step. 4 } \varkappa_{i+1} = o_i - \frac{\phi(o_i)}{j'(o_i)} \end{aligned}\right\} \tag{8}$$

Finally, (8) is our proposed Scheme, requiring only one first derivative evaluation and four function evaluations per iteration.

3. CONVERGENCE ANALYSIS

Theorem I: Consider α belonging to the set D, and let it be a simple root of a sufficiently differentiable function $\phi: D \subset \mathbb{R} \to \mathbb{R}$. Here, D represents an open interval that includes $\varkappa_0$ as an initial estimate of α. In such circumstances, the method outlined in equation (8) achieves a fifteenth-order accuracy. Remarkably, it required only four function evaluations and a single first derivative calculation in each complete iteration, eliminating the necessity for second or higher-order derivatives.

Proof.
The function $\phi(\varkappa_i)$ can be expressed using its Taylor series expansion as:

$$\phi(\varkappa_i) = \sum_{m=0}^{\infty} \frac{\phi^m(\alpha)}{m!}(\varkappa_i - \alpha)^m = \phi(\alpha) + \phi'(\alpha)(\varkappa_i - \alpha) + \frac{\phi''(\alpha)}{2!}(\varkappa_i - \alpha)^2 + \frac{\phi'''(\alpha)}{3!}(\varkappa_i - \alpha)^3 + \cdots$$

For simplicity, we assume that

$$C_k = \left(\frac{1}{k!}\right)\frac{\phi^k(\alpha)}{\phi'(\alpha)}, k \geq 2.$$

and assume that $\omega_i = \varkappa_i - \alpha$. Thus, we have:
For step one:

$$\phi(\varkappa_i) = \phi'(\alpha)\left(\omega_i + C_2\omega_i^2 + C_3\omega_i^3 + C_4\omega_i^4 + \cdots + O\left(\omega_i^{16}\right)\right) \tag{9}$$

and

$$\phi'(\varkappa_i) = \phi'(\alpha)\left(1 + 2C_2\omega_i + 3C_3\omega_i^2 + 4C_4\omega_i^3 + \cdots + O\left(\omega_i^{15}\right)\right) \tag{10}$$

From eq (9) and (10)

$$\frac{\phi(\varkappa_i)}{\phi'(\varkappa_i)} = \frac{\phi'(\alpha)\begin{pmatrix}\omega_i + C_2\omega_i^2 + C_3\omega_i^3 + C_4\omega_i^4 + C_5\omega_i^5 \\ +C_6\omega_i^6 + C_7\omega_i^7 + \cdots + O(\omega_i^{16})\end{pmatrix}}{\phi'(\alpha)\begin{pmatrix}1 + 2C_2\omega_i + 3C_3\omega_i^2 + 4C_4\omega_i^3 + 5C_5\omega_i^4 \\ +6C_6\omega_i^5 + 7C_7\omega_i^6 + \cdots + O(\omega_i^{15})\end{pmatrix}} \tag{11}$$

and

$$\frac{\phi(\varkappa_i)}{\phi'(\varkappa_i)} = \omega_i - C_2\omega_i^2 + (2C_2^2 - 2C_3)\omega_i^3 + (-4C_2^3 + 7C_3C_2 - 3C_4)\omega_i^4 + (8C_2^4 - 20C_3C_2^2 + 10C_4C_2 + 6C_3^2 - 4C_5)\omega_i^5 + \cdots + O(\omega_i^{16}) \tag{12}$$

Step. 1 $$v_i = \varkappa_i - \frac{\phi(\varkappa_i)}{\phi'(\varkappa_i)} = C_2\omega_i^2 + (2C_3 - 2C_2^2)\omega_i^3 + (4C_2^3 - 7C_3C_2 + 3C_4)\omega_i^4 + (-8C_2^4 + 20C_3C_2^2 - 10C_4C_2 - 6C_3^2 + 4C_5)\omega_i^5 + \cdots + O(\omega_i^{16}) \tag{13}$$

Now for step two:

$$\phi(v_i) = \phi'(\alpha)\begin{pmatrix}C_2\omega_i^2 + (2C_3 - 2C_2^2)\omega_i^3 + (5C_2^3 - 7C_3C_2 + 3C_4)\omega_i^4 \\ -2(6C_2^4 - 12C_3C_2^2 + 5C_4C_2 + 3C_3^2 - 2C_5)\omega_i^5 + \cdots + O\left(\omega_i^{16}\right)\end{pmatrix} \tag{14}$$

From equation (14) and (10):

$$\frac{\phi(v_i)}{\phi'(\varkappa_i)} = \frac{\phi'(\alpha)\begin{pmatrix}C_2\omega_i^2 + (2C_3 - 2C_2^2)\omega_i^3 \\ +(4C_2^3 - 7C_3C_2 + 3C_4)\omega_i^4 + \cdots + O(\omega_i^{16})\end{pmatrix}}{\phi'(\alpha)\begin{pmatrix}1 + 2C_2\omega_i + 3C_3\omega_i^2 + 4C_4\omega_i^3 + 5C_5\omega_i^4 \\ +6C_6\omega_i^5 + 7C_7\omega_i^6 + \cdots + O(\omega_i^{15})\end{pmatrix}} \tag{15}$$

After simplifying we get:

$$\frac{\phi(v_i)}{\phi'(\varkappa_i)} = C_2\omega_i + (2C_3 - 3C_2^2)\omega_i^2 + (8C_2^3 - 10C_3C_2 + 3C_4)\omega_i^3 + (-20C_2^4 + 37C_3C_2^2 - 14C_4C_2 - 8C_3^2 + 4C_5)\omega_i^4 + \left(48C_2^5 - 118C_3C_2^3 + 51C_4C_2^2 + 55C_3^2C_2 - 18C_5C_2 - 22C_3C_4 + 5C_6\right)\omega_i^5 + \cdots + O\left(\omega_i^{16}\right) \tag{16}$$

And weight function:

$$P = 1 + 2a + a^2 + a^3, \text{ and } a = \frac{\phi(v_i)}{\phi(\varkappa_i)} \tag{17}$$

$$P = -1 - 2C_2\omega_i + (5C_2^2 + C_2 - 4C_3)\omega_i^2 + (-11C_2^3 - 2C_2^2 + 16C_3C_2 + 2C_3 - 6C_4)\omega_i^3 + (24C_2^4 + 4C_2^3 - 48C_3C_2^2 - 7C_3C_2 + 22C_4C_2 + 12C_3^2 + 3C_4 - 8C_5)\omega_i^4 + \cdots + O\left(\omega_i^{16}\right) \tag{18}$$

$$P\frac{\phi(v_i)}{\phi'(\varkappa_i)} = C_2\omega_i^2 + (2C_3 - 2C_2^2)\omega_i^3 - 3(2C_2C_3 - C_4)\omega_i^4 + (19C_2^4 - 6C_3C_2^2 - 8C_4C_2 - 4C_3^2 + 4C_5)\omega_i^5 \ldots + O(\omega_i^{16}) \tag{19}$$

Step. 2 $$\xi_i = v_i - P\left(\frac{\phi(v_i)}{\phi'(v_i)}\right) = (4C_2^3 - C_2C_3)\omega_i^4 + (-27C_2^4 + 26C_3C_2^2 - 2C_4C_2 - 2C_3^2)\omega_i^5 + \cdots + O\left(\omega_i^{16}\right) \tag{20}$$

$$\phi(\xi_i) = \phi'(\alpha)\left((4C_2^3 - C_2C_3)\omega_i^4 + \begin{pmatrix}-27C_2^4 + 26C_3C_2^2 \\ -2C_4C_2 - 2C_3^2\end{pmatrix}\omega_i^5 + \cdots + O\left(\omega_i^{16}\right)\right) \tag{21}$$

$$H'(\xi_i) = 2(\phi[\varkappa_i,\xi_i] - \phi[v_i,\varkappa_i]) + \phi[v_i,\xi_i] + (v_i - \xi_i)\phi[v_i,\varkappa_i,\varkappa_i] \tag{22}$$

$$H_3'(\xi_i) = \phi'(\alpha)\begin{pmatrix}1 + C_2(8C_2^3 - 2C_3C_2 + C_4)\omega_i^4 \\ -2\begin{pmatrix}27C_2^5 - 26C_3C_2^3 + 3C_4C_2^2 \\ +2C_3^2C_2 - C_5C_2 - C_3C_4\end{pmatrix}\omega_i^5 + \cdots + O\left(\omega_i^{16}\right)\end{pmatrix} \tag{23}$$

$$\frac{\phi(\xi_i)}{H_3'(\xi_i)} = \frac{\phi'(\alpha)\begin{pmatrix} (4C_2^3 - C_2C_3)\omega_i^4 \\ +\begin{pmatrix} -27C_2^4 + 26C_3C_2^2 \\ -2C_4C_2 - 2C_3^2 \end{pmatrix}\omega_i^5 + \cdots + O(\omega_i^{16}) \end{pmatrix}}{\phi'(\alpha)\begin{pmatrix} 1 + C_2(8C_2^3 - 2C_3C_2 + C_4)\omega_i^4 \\ -2\begin{pmatrix} 27C_2^5 - 26C_3C_2^3 + 3C_4C_2^2 \\ +2C_3^2C_2 - C_5C_2 - C_3C_4 \end{pmatrix}\omega_i^5 + \cdots + O(\omega_i^{16}) \end{pmatrix}} \tag{24}$$

$$\frac{\phi(\xi_i)}{H_3'(\xi_i)} = (4C_2^3 - C_2C_3)\omega_i^4 + \begin{pmatrix} -27C_2^4 + 26C_3C_2^2 \\ -2C_4C_2 - 2C_3^2 \end{pmatrix}\omega_i^5 + \begin{pmatrix} 121C_2^5 - 196C_3C_2^3 + 39C_4C_2^2 \\ +54C_3^2C_2 - 3C_5C_2 - 7C_3C_4 \end{pmatrix}\omega_i^6 + \cdots + O(\omega_i^{16}) \tag{25}$$

Step. 3 $o_i = \xi_i - \frac{\phi(\xi_i)}{H'(\xi_i)} = \begin{pmatrix} 16C_2^7 - 8C_3C_2^5 + 4C_4C_2^4 \\ +C_3^2C_2^3 - C_3C_4C_2^2 \end{pmatrix}\omega_i^8$

$$+\begin{pmatrix} -216C_2^8 + 262C_3C_2^6 - 51C_4C_2^5 - 68C_3^2C_2^4 + 8C_5C_2^4 \\ +40C_3C_4C_2^3 + 4C_3^3C_2^2 - 2C_4^2C_2^2 - 2C_3C_5C_2^2 - 4C_3^2C_4C_2 \end{pmatrix}\omega_i^9 + \cdots + O(\omega_i^{16}) \tag{26}$$

$$\phi(o_i) = \phi'(\alpha)\begin{pmatrix} \begin{pmatrix} 16C_2^7 - 8C_3C_2^5 + 4C_4C_2^4 \\ +C_3^2C_2^3 - C_3C_4C_2^2 \end{pmatrix}\omega_i^8 \\ +\begin{pmatrix} -216C_2^8 + 262C_3C_2^6 - 51C_4C_2^5 - 68C_3^2C_2^4 + 8C_5C_2^4 \\ +40C_3C_4C_2^3 + 4C_3^3C_2^2 - 2C_4^2C_2^2 - 2C_3C_5C_2^2 - 4C_3^2C_4C_2 \end{pmatrix}\omega_i^9 + \cdots + O(\omega_i^{16}) \end{pmatrix} \tag{27}$$

$$j'(o_i) = \phi[\varkappa_i, \xi_i] + (\phi[v_i, \varkappa_i, \xi_i] - \phi[v_i, \varkappa_i, o_i] - \phi[\xi_i, \varkappa_i, o_i])(v_i - \xi_i) \tag{28}$$

$$j'(o_i) = \phi'(\alpha)\begin{pmatrix} 1 + C_2^2(4C_2^2 - C_3)C_4\omega_i^7 \\ +C_2\begin{pmatrix} 32C_2^7 - 16C_3C_2^5 - 27C_4C_2^4 + 2(C_3^2 + 2C_5)C_2^3 \\ +34C_3C_4C_2^2 - (2C_4^2 + C_3C_5)C_2 - 4C_3^2C_4 \end{pmatrix}\omega_i^8 + \cdots + O(\omega_i^{16}) \end{pmatrix} \tag{29}$$

$$\frac{\phi(o_i)}{j'(o_i)} = \frac{\phi'(\alpha)\begin{pmatrix} \begin{pmatrix} 16C_2^7 - 8C_3C_2^5 + 4C_4C_2^4 \\ +C_3^2C_2^3 - C_3C_4C_2^2 \end{pmatrix}\omega_i^8 \\ +\begin{pmatrix} -216C_2^8 + 262C_3C_2^6 - 51C_4C_2^5 \\ -68C_3^2C_2^4 + 8C_5C_2^4 + 40C_3C_4C_2^3 \\ +4C_3^3C_2^2 - 2C_4^2C_2^2 - 2C_3C_5C_2^2 \\ -4C_3^2C_4C_2 \end{pmatrix}\omega_i^9 + \cdots + O(\omega_i^{16}) \end{pmatrix}}{\phi'(\alpha)\begin{pmatrix} 1 + C_2^2(4C_2^2 - C_3)C_4\omega_i^7 \\ +C_2\begin{pmatrix} 32C_2^7 - 16C_3C_2^5 - 27C_4C_2^4 \\ +2(C_3^2 + 2C_5)C_2^3 + 34C_3C_4C_2^2 \\ -(2C_4^2 + C_3C_5)C_2 - 4C_3^2C_4 \end{pmatrix}\omega_i^8 + \cdots + O(\omega_i^{16}) \end{pmatrix}} \tag{30}$$

$$\frac{\phi(o_i)}{j'(o_i)} = C_2^2(4C_2^2 - C_3)\begin{pmatrix} 4C_2^3 \\ -C_3C_2 + C_4 \end{pmatrix}\omega_i^8 - C_2\begin{pmatrix} 216C_2^7 - 262C_3C_2^5 + 51C_4C_2^4 \\ +(68C_3^2 - 8C_5)C_2^3 - 40C_3C_4C_2^2 \\ +2(-2C_3^3 + C_5C_3 + C_4^2)C_2 + 4C_3^2C_4 \end{pmatrix}\omega_i^9 + \cdots + O(\omega_i^{16}) \tag{31}$$

Step. 4

$$\varkappa_{i+1} = o_i - \frac{\phi(o_i)}{j'(o_i)} = C_2^4(C_3 - 4C_2^2)^2C_4(4C_2^3 - C_3C_2 + C_4)\omega_i^{15} + O(\omega_i^{16}) \tag{32}$$

Finally, the proposed scheme mentioned in (8) exhibits a convergence rate of fifteen, and the efficiency index for proposed algorithm is calculated to be 1.718771928.

4. NUMERICAL EXPRIMENTS

The problems presented below were sourced from literature [12, 13] and assessed using both the Proposed Scheme and its corresponding counterpart. Tables and graphs illustrate the Computational Order of Convergence (COC) [14], accuracy, efficiency, and consistency of the Proposed Scheme and its analogous methods. All the problems below were solved using Maple 2022 software, and line plots were generated using Origin 2021 software on my personal laptop with the following specifications: Intel® Core™ i3-4010U CPU @ 1.70 GHz and 8.00 GB RAM.

Problem 1.

$$\phi_1(\varkappa) = \sin\left(\frac{1}{\varkappa}\right) - \varkappa$$
$$\phi_2(\varkappa) = 1 + e^{\varkappa^2 + \varkappa} - \cos(-\varkappa^2 + 1) + \varkappa^3$$
$$\phi_3(\varkappa) = (\varkappa - 1)^2 - 1$$
$$\phi_4(\varkappa) = \sqrt{\varkappa^2 + 2\varkappa + 5} - 2\sin(\varkappa) - \varkappa^2 + 3$$
$$\phi_5(\varkappa) = 10\varkappa e^{-\varkappa^2} - 1$$
$$\phi_6(\varkappa) = \cos^2(\varkappa) - \frac{\varkappa}{5}$$

Figure 1 shows the comparison of the Computational Order of Convergence (COC) for the proposed method and its counterpart methods for Problem 1. In this figure, subfigures (1a), (1b), (1c), (1d), (1e) and (1f) display the graphical representations of $\phi_1(\varkappa), \phi_2(\varkappa), \phi_3(\varkappa), \phi_4(\varkappa), \phi_5(\varkappa)$ and $\phi_6(\varkappa)$ respectively. Above all figures are shown that proposed method converges faster as compared to counterparts.

Table 1. Value of $|\varkappa_1 - \varkappa_0|, |\varkappa_2 - \varkappa_1|, |\varkappa_3 - \varkappa_2|$ & COC of different methods of order fifteenth of problems $\phi_1(\varkappa)$ to $\phi_6(\varkappa)$.

$(\phi_n(\varkappa), \varkappa_0)$	Proposed scheme	TE1 15th	TE2 15th	TE3 15th	FS 15th
			$(\phi_1(\varkappa), -0.01)$		
$\|\varkappa_1 - \varkappa_0\|$	$5.3800e-5$	$5.3800e-5$	$5.3800e-5$	$5.3800e-5$	$5.3800e-5$
$\|\varkappa_2 - \varkappa_1\|$	$1.7033e-16$	$6.3242e-16$	$2.5633e-16$	$7.2405e-16$	$2.9040e-16$
$\|\varkappa_3 - \varkappa_2\|$	$9.9051e-197$	$6.1561e-185$	$1.7937e-194$	$2.3415e-184$	$9.8668e-194$
COC	15.773	15.463	15.735	15.499	15.750
			$(\phi_2(\varkappa), 0.3)$		
$\|\varkappa_1 - \varkappa_0\|$	1.2457 ...	*Diverge*	*Diverge*	*Diverge*	*Diverge*
$\|\varkappa_2 - \varkappa_1\|$	$5.4320e-2$	*Diverge*	*Diverge*	*Diverge*	*Diverge*
$\|\varkappa_3 - \varkappa_2\|$	$1.4298e-23$	*Diverge*	*Diverge*	*Diverge*	*Diverge*
COC	15.862	*Diverge*	*Diverge*	*Diverge*	*Diverge*
			$(\phi_3(\varkappa), 0.1)$		
$\|\varkappa_1 - \varkappa_0\|$	1.9000 ...	*Diverge*	*Diverge*	*Diverge*	2.2048 ...
$\|\varkappa_2 - \varkappa_1\|$	$2.8218e-5$	*Diverge*	*Diverge*	*Diverge*	$3.0475e-1$
$\|\varkappa_3 - \varkappa_2\|$	$2.9192e-71$	*Diverge*	*Diverge*	*Diverge*	$3.3542e-10$
COC	13.667	*Diverge*	*Diverge*	*Diverge*	10.424
			$(\phi_4(\varkappa), 1.2)$		
$\|\varkappa_1 - \varkappa_0\|$	1.1320 ...	1.1320 ...	1.1320 ...	1.1320 ...	1.1320 ...
$\|\varkappa_2 - \varkappa_1\|$	$4.4546e-14$	$1.3539e-13$	$5.0004e-14$	$4.7627e-14$	$4.3702e-14$
$\|\varkappa_3 - \varkappa_2\|$	$8.0368e-211$	$2.1381e-202$	$9.8294e-210$	$9.6175e-210$	$8.8286e-211$
COC	14.677	14.611	14.654	14.630	14.664
			$(\phi_5(\varkappa), -0.1)$		
$\|\varkappa_1 - \varkappa_0\|$	$2.0103e-1$	$2.0103e-1$	$2.0103e-1$	$2.0103e-1$	$2.0103e-1$
$\|\varkappa_2 - \varkappa_1\|$	$1.3984e-16$	$1.3984e-16$	$4.8642e-16$	$1.2235e-14$	$1.7467e-16$
$\|\varkappa_3 - \varkappa_2\|$	$1.0390e-241$	$1.8213e-216$	$1.2218e-232$	$3.5682e-212$	$1.1647e-239$
COC	14.853	13.187	14.819	14.471	14.818
			$(\phi_6(\varkappa), 0.1)$		
$\|\varkappa_1 - \varkappa_0\|$	2.2145 ...	*Diverge*	3.8930 ...	*Diverge*	1.4335 ...
$\|\varkappa_2 - \varkappa_1\|$	$5.6587e-3$	*Diverge*	$3.1198e-1$	*Diverge*	$4.5480e-1$
$\|\varkappa_3 - \varkappa_2\|$	$2.1410e-41$	*Diverge*	$2.0877e-14$	*Diverge*	$7.2712e-3$
COC	14.820	*Diverge*	12.019	*Diverge*	3.6026

Problem 2. The van der Waals equation see in Liu and Lee [15].

$$\phi(\varkappa) = 0.986\varkappa^3 - 5.181\varkappa^2 + 9.067\varkappa - 5.289 \tag{33}$$

Problem 3. The governs the depth of embedment equation See in Ding *et al.* [16].

$$\phi(\varkappa) = \frac{\varkappa^3 + 2.87\varkappa^2 - 4.62\varkappa - 10.28}{4.62} \tag{34}$$

Problem 4. The multipactor phenomenon See in Naseem *et al.* [17].

$$\phi(\varkappa) = \varkappa - \frac{1}{2}cos(\varkappa) + \frac{\pi}{4} \tag{35}$$

4.1. Description of Basin of Attraction

The examination of solution stability for the nonlinear function $\phi(\varkappa) = 0$, achieved through an Iterative Scheme, can be simplified using the concept of basins of attraction [20, 29]. Utilizing MATLAB R2014a, a visual depiction of all basins within the range $R = [-5 \times 5] \times [-5 \times 5]$ was created, consisting of $360{,}000$ points at a density of 600×600.

For iteration termination criteria, two conditions were set: a maximum iteration count of 10 or an error threshold of 1×10^{-10}. Each point within R initiated the iterative algorithms. If, within 10 iterations, the sequence from the iterative algorithm converged to a root $\varkappa_k^*$ of the

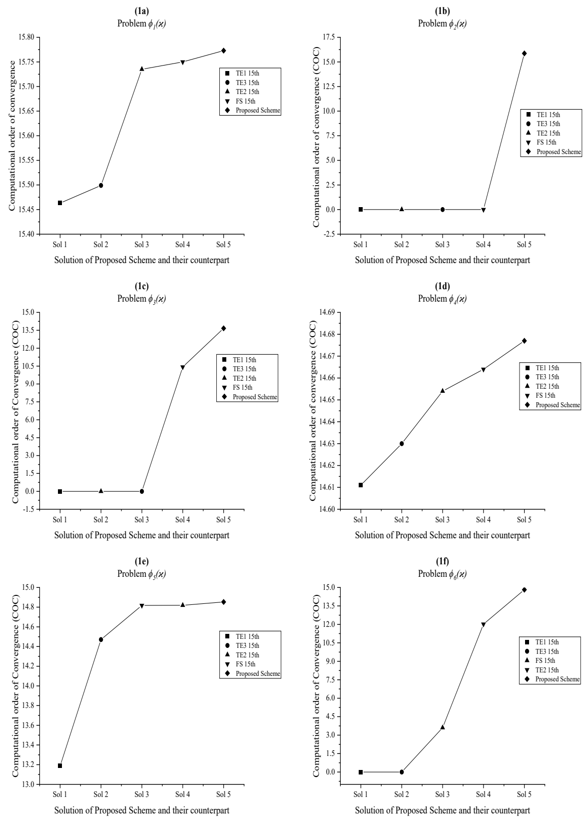

Fig. 1. COC versus solution of problem $\phi_1(\varkappa)$ to $\phi_6(\varkappa)$ by Proposed Scheme and their counterparts.

function $P_k(\varkappa)$ and met the specified tolerance, a distinct color (excluding black) was assigned to the starting point, corresponding to the convergent root. Conversely, if the iterative algorithm initiated with a point $\varkappa \in R$ exceeded the maximum iteration count of 16 without converging to any root $\varkappa_k$ or converging to a different value p within the specified tolerance $|p - \varkappa^*| < 1 \times 10^{-10}$, it was determined that the starting point had diverged. In such instances, a black color was assigned to the starting point.

Problem 5. Below problems were taken from literature.

S. No.	Functions $(P(\varkappa))$	Roots $(\varkappa_k : k = 1,2,3,\ldots)$
i.	$P_1(\varkappa) = \varkappa^5 + 1$	$\varkappa_k = -1, -\frac{305}{987} \pm \frac{855}{899}i, \frac{1292}{1597} \pm \frac{4456}{7581}i$
ii.	$P_2(\varkappa) = \varkappa^3 + 1$	$\varkappa_k = 1, \frac{1 \pm \sqrt{3}i}{2}$
iii.	$P_3(\varkappa) = \varkappa^2 + 2\varkappa - \frac{1}{2}$	$\varkappa_k = \frac{-2 \pm \sqrt{6}}{2}$
iv.	$P_4(\varkappa) = \varkappa^4 + \frac{1}{64}$	$\varkappa_k = \frac{1 \pm 1i}{4}, \frac{-1 \pm 1i}{4}$
v.	$P_5(\varkappa) = \varkappa^5 - \frac{1}{2}i\varkappa^4 + \frac{1}{64}\varkappa - \frac{1}{128}i$	$\varkappa_k = \frac{1 \pm 1i}{4}, \frac{-1 \pm 1i}{4}, \frac{1}{2}i$
vi.	$P_6(\varkappa) = \varkappa^2 - \frac{1}{4}$	$\varkappa_k = \frac{1}{2}, -\frac{1}{2}$

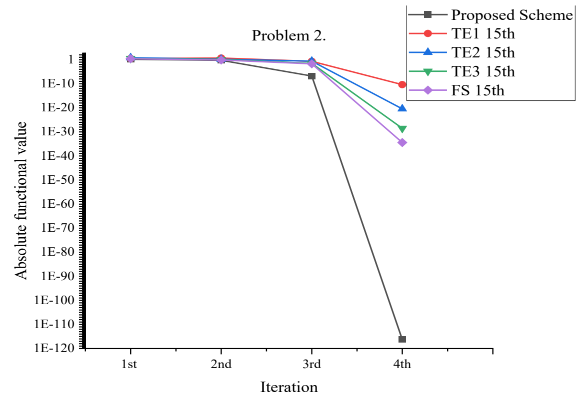

Fig. 2. Solution versus iterations of problem 2. by Proposed Scheme and its counterparts by assuming the scale $1E - 30 = 1E - 1$.

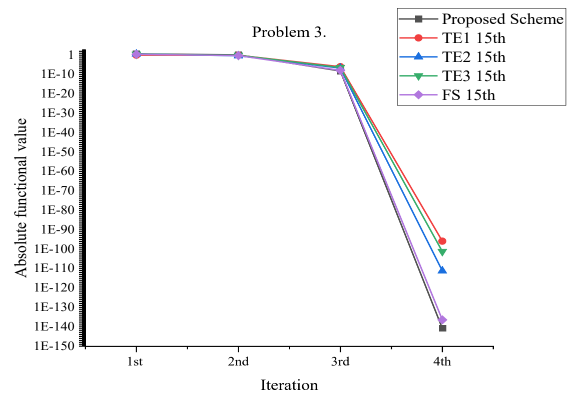

Fig. 3. Solution versus iterations of problem 3. by Proposed Scheme and its counterparts by assuming the scale $1E - 300 = 1E - 1$.

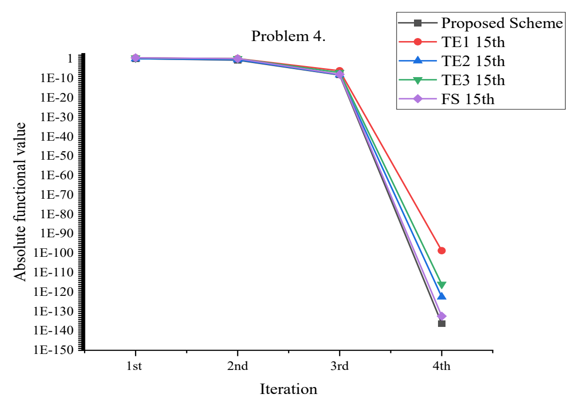

Fig. 4. Solution versus iterations of problem 4. by Proposed Scheme and its counterparts by assuming the scale $1E - 200 = 1E - 1$.

Figure 5 illustrates the basins of attraction for Problem 5, with all initial guesses within the range $R = [-5 \times 5] \times [-5 \times 5]$, as obtained by the proposed method. Subfigures (5a) and (5b) correspond to the basin of attraction for $P_1(\varkappa)$, which has five roots, with (5a) illustrating the root

Table 2. Numerical results for problem 2. for first four iterations and their absolute functional values at $\varkappa_0 = 1.5$.

Method	Root & corresponding absolute functional value	1st iteration	2nd iteration	3rd iteration	4th iteration
Proposed scheme	$\varkappa$	1.9482 ...	1.9298 ...	1.9298 ...	1.9298 ...
	$\lvert\phi(\varkappa)\rvert$	$1.7732E-3$	$1.9421E-16$	$4.1241E-222$	$7.0501E-3515$
TE1 15th	$\varkappa$	2.4548 ...	1.9373 ...	1.9298 ...	1.9298 ...
	$\lvert\phi(\varkappa)\rvert$	$3.3344E-1$	$6.8210E-4$	$1.2382E-23$	$1.8522E-319$
TE2 15th	$\varkappa$	2.1918 ...	1.9301 ...	1.9298 ...	1.9298 ...
	$\lvert\phi(\varkappa)\rvert$	$7.6603E-2$	$2.3684E-5$	$7.9528E-45$	$6.3774E-637$
TE3 15th	$\varkappa$	2.1120 ...	1.9298 ...	1.9298 ...	1.9298 ...
	$\lvert\phi(\varkappa)\rvert$	$3.9203E-2$	$1.8729E-6$	$3.1684E-61$	$8.4513E-883$
FS 15th	$\varkappa$	2.0851 ...	1.9298 ...	1.9298 ...	1.9298 ...
	$\lvert\phi(\varkappa)\rvert$	$2.9826E-2$	$1.2588E-6$	$3.2506E-67$	$1.2738E-1036$

Table 3. Numerical results for problem 3. for first four iterations and their absolute function values at $\varkappa_0 = -0.5$.

Method	Root & corresponding absolute functional value	1st iteration	2nd iteration	3rd iteration	4th iteration
Proposed scheme	$\varkappa$	−1.5417 ...	−1.5417 ...	−1.5417	−1.5417 ...
	$\lvert\phi(\varkappa)\rvert$	$5.6042E-10$	$5.6042E-166$	$4.7846E-2652$	$3.8128E-42429$
TE1 15th	$\varkappa$	−1.5417 ...	−1.5417 ...	−1.5417 ...	−1.5417 ...
	$\lvert\phi(\varkappa)\rvert$	$1.1165E-8$	$2.4491E-128$	$3.2057E-1923$	$1.8188E-28846$
TE2 15th	$\varkappa$	−1.5417 ...	−1.5417 ...	−1.5417 ...	−1.5417 ...
	$\lvert\phi(\varkappa)\rvert$	$4.2762E-10$	$1.5897E-148$	$5.6951E-2225$	$1.1695E-33371$
TE3 15th	$\varkappa$	−1.5417 ...	−1.5417 ...	−1.5417 ...	−1.5417 ...
	$\lvert\phi(\varkappa)\rvert$	$3.1795E-9$	$2.7183E-135$	$2.5914E-2026$	$1.2648E-30391$
FS 15th	$\varkappa$	−1.5417 ...	−1.5417 ...	−1.5417 ...	−1.5417 ...
	$\lvert\phi(\varkappa)\rvert$	$3.0162E-10$	$2.0774E-160$	$5.3247E-2563$	$1.8479E-41004$

Table 4. Numerical results for problem 4. for first four iterations and their absolute function values at $\varkappa_0 = 2.3$.

Method	Root & corresponding absolute functional value	1st iteration	2nd iteration	3rd iteration	4th iteration
Proposed scheme	$\varkappa$	−0.3090 ...	−0.3090 ...	−0.3090 ...	−0.3090 ...
	$\lvert\phi(\varkappa)\rvert$	$1.9167E-8$	$1.1780E-121$	$7.9496E-1820$	$2.1777E-27292$
TE1 15th	$\varkappa$	−0.3090 ...	−0.3090 ...	−0.3090 ...	−0.3090 ...
	$\lvert\phi(\varkappa)\rvert$	$4.3835E-6$	$5.0330E-88$	$3.9988E-1317$	$1.2690E-19753$
TE2 15th	$\varkappa$	−0.3090 ...	−0.3090 ...	−0.3090 ...	−0.3090 ...
	$\lvert\phi(\varkappa)\rvert$	$1.6766E-7$	$1.6490E-109$	$1.2856E-1639$	$3.0731E-24591$
TE3 15th	$\varkappa$	−0.3090 ...	−0.3090 ...	−0.3090 ...	−0.3090 ...
	$\lvert\phi(\varkappa)\rvert$	$4.0270E-7$	$5.5846E-104$	$7.5362E-1557$	$6.7531E-23350$
FS 15th	$\varkappa$	−0.3090 ...	−0.3090 ...	−0.3090 ...	−0.3090 ...
	$\lvert\phi(\varkappa)\rvert$	$4.8649E-8$	$6.7560E-118$	$9.3098E-1766$	$1.1420E-26483$

locations and (5b) depicting the number of iterations required for convergence. Subfigures (5c) and (5d) show the basin of attraction for $P_2(\varkappa)$ with three roots, where (5c) shows the root positions and (5d) represents the iteration count for each initial guess. For $P_3(\varkappa)$ with two roots, subfigures (5e) and (5f) displaying the roots and the required iterations, respectively. Subfigures (5g) and (5h) represent the basin of attraction for $P_4(\varkappa)$ with four roots, (5i) and (5j) for $P_5(\varkappa)$with five roots, and (5k) and (5l) for $P_6(\varkappa)$ with two roots. In each pair of subfigures, the left panel visualizes the location of the roots, while the right panel indicates the number of iterations required to achieve convergence to the respective root.

5. SUMMARY

To summarize, it is observed that TE1 15th diverges in problem 1 of $\phi_2(\varkappa)$, $\phi_3(\varkappa)$, and $\phi_6(\varkappa)$, TE2 15th diverges in $\phi_2(\varkappa)$ and $\phi_3(\varkappa)$, TE3 15th diverges in $\phi_2(\varkappa)$, $\phi_3(\varkappa)$and $\phi_6(\varkappa)$, while FS 15th diverges in $\phi_2(\varkappa)$, as detailed in Table 1. Both Table 1 and Figure 1 demonstrate that the Computational Order of Convergence (COC) of the Proposed Scheme exceeds that of all other methods across all test problems. Additionally, real-world problems 2-4, presented in Tables 2-4 and Figures 2-4, indicate that the Proposed Scheme achieves faster convergence compared to alternative methods. The stability analysis of the Proposed Scheme is depicted in Figure 5 through the basin of attraction.

6. CONCLUSIONS

The research paper introduces a new method for solving non-linear algebraic and transcendental equations, utilizing a weight function and interpolations. Key features include a fifteenth-order convergence rate, requiring only four function evaluations and one first derivative per iteration. Analysis of test problems reveals divergence patterns in TE1 15th, TE2 15th, TE3 15th, and FS 15th. Notably, the Proposed Scheme demonstrates superior computational Order of Convergence (COC) compared to counterparts in Table 1. Real-world problems 1-3 exhibit the proposed scheme's rapid convergence relative to other methods, and stability analysis in Figure 5 confirms its stability. Overall, the Proposed Scheme proves efficient, fast convergent, stable, and consistent, representing a significant advancement in the literature.

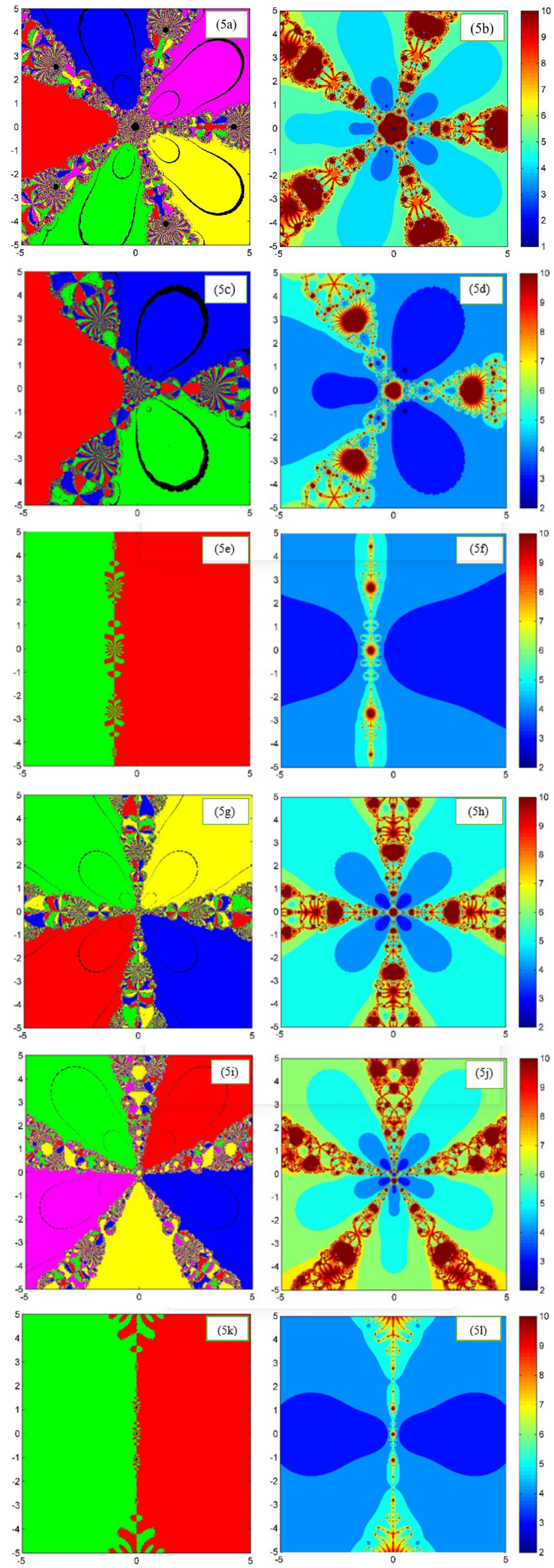

Fig. 5. Basin of attraction of $P_1(\varkappa)$ to $P_6(\varkappa)$ of problems 5 obtained by the proposed method.

7. ACKNOWLEDGEMENT

The authors express their gratitude to Inistitute of Mathematics and Computer Science, University of Sindh, Jamshoro, Sindh, for facilitating the completion of this research.

8. CONFLICT OF INTEREST

The authors declare no conflict of interest.

9. REFERENCES

1. S. Thota and P. Shanmugasundaram. On new sixth and seventh order iterative methods for solving non-linear equations using homotopy perturbation technique. *BMC Research Notes* 15(1): 1-15 (2022).
2. Z. Xiaojian. Modified Chebyshev-Halley methods free from second derivative. *Applied Mathematics and Computation* 203(2): 824-827 (2008).
3. F.A. Lakho, Z.A. Kalhoro, S. Jamali, A.W. Shaikh, and J. Guan. A three steps seventh order iterative method for solution nonlinear equation using Lagrange Interpolation technique. *VFAST Transactions on Mathematics* 12(1): 46-59 (2024).
4. A. Naseem, M.A. Rehman, and J. Younis. A New Root-Finding Algorithm for Solving Real-World Problems and Its Complex Dynamics via Computer Technology. *Complexity* 2021: 1-10 (2021).
5. Z. Abbasi, Z.A. Kalhoro, S. Jamali, A.W. Shaikh, and O.A. Rajput. A novel Approach for Real-World Problems Based on Hermite Interpolation Technique and Analysis Using Basins of Attraction. *The Sciencetech.* 5(3): 112-126 (2024).
6. M.I. Soomro, Z.A. Kalhoro, A.W. Shaikh, S. Jamali, and O. Ali. Modified Bracketing Iterative Method for Solving Nonlinear Equations. *VFAST Transactions on Mathematics* 12(1): 105-120 (2024).
7. T. Eftekhari. A new family of four-step fifteenth-order root-finding methods with high efficiency index. *Computational Methods for Differential Equations* 3(1): 51-58 (2015).
8. F. Soleymani and M. Sharifi. On a class of fifteenth-order iterative formulas for simple roots. *International Electronic Journal of Pure and Applied Mathematics* 3(3): 245-252 (2011).
9. S. Jamali, Z.A. Kalhoro, A.W. Shaikh, and M.S. Chandio. An iterative, bracketing & derivative-free method for numerical solution of non-linear equations using stirling interpolation technique. *Journal of Mechanics of Continua and Mathematical Sciences* 16(6): 13-27 (2021).
10. Z. Kovacs. Understanding convergence and stability of the Newton-Raphson method. *Teaching Mathematics and Statistics in Sciences*: 1-12 (2011). http://www.model.u-szeged.hu/data/etc/edoc/imp/ZKovacs/ZKovacs.pdf.
11. M. Kumar, A.K. Singh, and A. Srivastava. Various Newton-type iterative methods for solving nonlinear equations. *Journal of the Egyptian Mathematical Society* 21(3): 334-339 (2013).
12. S. Jamali, Z.A. Kalhoro, A.W. Shaikh, M.S. Chandio, and S. Dehraj. A novel two point optimal derivative free method for numerical solution of nonlinear algebraic, transcendental Equations and application problems using weight function. *VFAST Transactions on Mathematics* 10(2): 137-146 (2022).
13. P. Sivakumar and J. Jayaraman. Some New Higher Order Weighted Newton Methods for Solving Nonlinear Equation with Applications. *Mathematical and Computational Applications* 24(2): 1-16 (2019).
14. M. Grau-Sánchez, M. Noguera, À. Grau, and J.R. Herrero. On new computational local orders of convergence. *Applied Mathematics Letters* 25(12): 2023-2030 (2012).
15. C.S. Liu and T.L. Lee. A New Family of Fourth-Order Optimal Iterative Schemes and Remark on Kung and Traub's Conjecture. *Journal of Mathematics* 2021: 5516694 (2021).
16. H. fei Ding, Y. xin Zhang, S. fu Wang, and X. ya Yang. A note on some quadrature based three-step iterative methods for non-linear equations. *Applied Mathematics and Computation* 215(1): 53-57 (2009).
17. A. Naseem, M.A. Rehman, and T. Abdeljawad. Computational methods for non-linear equations with some real-world applications and their graphical analysis. *Intelligent Automation and Soft Computing* 30(3): 805-819 (2021).