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Research Article

Coefficient Bounds for Certain Classes of Analytic Functions of Complex Order γ Associated with Cardioid Domain

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Abstract: In this paper, new Ma–Minda type subclasses of analytic functions of complex order γ, associated with the cardioid domain in the open unit disk $\mathbb{E}$, are introduced. The objective of this work is to investigate the first three sharp coefficient bounds and to examine the sharp estimates of the Fekete-Szegö problem for these subclasses.

Keywords: Analytic Functions, Starlike and Convex Functions, Cardioid Domain, Subordination, Coefficient Bounds, Fekete-Szegö Problem.

1. INTRODUCTION

In Geometric Function Theory, the classes $\mathcal{S}^*$ and $\mathcal{C}$, consisting of starlike and convex univalent functions respectively, are considered as the most extensively investigated subclasses of the class $\mathcal{S}$ of univalent functions in the open unit disk $\mathbb{E}=\{z\in\mathbb{C}:|z|<1\}$, and have been of major interest. We refer the reader to [1-5] for the basic theory of univalent functions, and to Duren [6] for a detailed review of starlike and convex functions. A central focus in this area is the investigation of coefficient problems and the Fekete-Szegö inequality, which provide deep insights into the behavior of such functions. For detailed discussions on the coefficient problem and the Fekete–Szegö inequality with related problems, see [7-9] and [10-12], respectively. To unify and extend the study of these and related subclasses, Ma–Minda form of analytic functions are used, which offers a unified approach to several subclasses of analytics functions through subordination techniques.

In this work, we introduce new Ma–Minda type subclasses of starlike and convex analytic functions connected with a cardioid domain and examine their geometric properties, including sharp coefficient bounds and the Fekete-Szegö problem. To proceed, we recall and review the following fundamental definitions, which will serve as the basis for our present study.

Let $\mathcal{A}$ be the class of analytic functions f, defined on $\mathbb{E}=\{z\in\mathbb{C}:|z|<1\}$, satisfying $f(0)=0$, $f'(0)=1$, and having the following form:

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n \quad (1)$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions f. Consider a subclass $\mathcal{P}\subset\mathcal{A}$, which includes functions with positive real part and admits the following power series expansion:

$$p(z)=1+\sum_{n=1}^{\infty}c_n z^n\,;\ \ (z\in\mathbb{E}) \quad (2)$$

Let f and g be analytic in the open unit disk. we say that f is subordinate to g, written as $f\prec g$, if there exists a function u, analytic in $\mathbb{E}$, satisying $u(0)=1$ and $|u(z)|<1$ such that $f(z)=g\big(u(z)\big)$. Moreover, if the function g is univalent in $\mathbb{E}$, then, $f\prec g\Leftrightarrow f(0)=g(0)$ and $f(\mathbb{E})\subset g(\mathbb{E})$.

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In 1992, Ma and Minda [13] using the concept of subordination, introduced the following suclasses of starlike and convex functions respectively.

$$S^*(\varphi) = \left\{ f \in \mathcal{A}: \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\} \qquad (3)$$

and

$$C(\varphi) = \left\{ f \in \mathcal{A}: 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\} \qquad (4)$$

Where the function $\varphi(z)$ satisfying $Re(\varphi(z)) > 0$, the image $\varphi(\mathbb{E})$ is symmetric with respect to the real axis, starlike with respect to the point $\varphi(0) = 1$ and $\varphi'(0) > 0$. It is worth noting that these classes encompass and unify many important subclasses of $\mathcal{A}$. For example, setting $\varphi(z) = \frac{1+A_1 z}{1+A_2 z}$ with $-1 \le A_2 < A_1 \le 1$, in (3) and (4), we obtained the renowned classes of Janowski starlike and convex functions, represented by $S^*[A_1, A_2]$ and $C[A_1, A_2]$, respectively, studied by Janowski [14]. In addition, if we take $A_1 = 1 - 2\propto$ and $A_2 = -1$ with $0 \le \propto < 1$, then $S^*(\varphi)$ and $C(\varphi)$ becomes $S^*(\propto)$ and $C(\propto)$, which are the classes of starlike and convex functions of order $\propto$ respectively, defined by Robertson [15]. The classes $S^*_{sin} = S^*(1 + sin\, z)$ and $S^*_{exp} = S^*(e^z)$ were studied by Cho *et al.* [16] and Aouf *et al.* [17], respectively. By choosing $\left(\frac{1+z}{1-z}\right)$ and $\sqrt{1+z}$ as specific forms of $\varphi(z)$, the class $S^*(\varphi)$ reduces to the classical starlike class S^* and the class S^*_L defined by Sokoł and Stankiewicz [18], respectively.

For $\varphi(z) = 1 + ze^z$, Kumar and Kamaljeet [19] introduced the subclass $\mathfrak{S}^*_{\wp}$ of of starlike functions defined by:

$$\mathfrak{S}^*_{\wp} = \left\{ f \in \mathcal{A}: \frac{zf'(z)}{f(z)} \prec 1 + ze^z =: \varphi(z) \right\} \qquad (5)$$

where the image of the unit disk under φ is a cardioid domain.

For $\varphi(z) = 1 + \frac{4z}{3} + \frac{2z^2}{3}$, the class reduces to S^*_C, investigated by Sharma *et al.* [20], containing functions $f \in \mathcal{A}$ such that $\frac{zf'(z)}{f(z)}$ mapped into the region enclosed by the cardioid described by the following Equation.

$$16(9x^2 + 9y^2 - 6x + 1) - (9x^2 + 9y^2 - 18x + 5)^2 = 0 \qquad (6)$$

An extension of (3) and (4) respectively, was given by Ravichandran *et al.* [21] as follows:

$$S^*(\gamma, \varphi) = \left\{ f \in \mathcal{A}: 1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \varphi(z) \right\} \qquad (7)$$

$$C(\gamma, \varphi) = \left\{ f \in \mathcal{A}: 1 + \frac{1}{\gamma}\left(\frac{zf''(z)}{f'(z)}\right) \prec \varphi(z) \right\} \qquad (8)$$

where, $\gamma \in \mathbb{C} \setminus \{0\}$.

These particular forms of functions are known as Ma–Minda type analytic functions of order γ, $(\gamma \in \mathbb{C} \setminus \{0\})$.

Recently, Al-Shaikh *et al.* [22] introduced the class $\mathcal{R}(l, m, \propto, \gamma)$, where $0 \le \propto \le 1$ and $\gamma \in \mathbb{C} \setminus \{0\}$, consisting of analytic functions $f \in \mathcal{A}$ of complex order γ, satisfying the subordination condition:

$$1 + \frac{1}{\gamma}\left(\frac{zf'(z) + \propto z^2 f''(z)}{(1-\propto)f(z) + \propto zf'(z)} - 1\right) \prec \varphi_{Car}(l, m; z)$$

where, $\varphi_{Car}(l, m; z)$ defines a cardioid domain given by:

$$\varphi_{Car}(l, m; z) = \frac{2l\tau^2 z^2 + (l-1)\tau z + 2}{2m\tau^2 z^2 + (m-1)\tau z + 2}$$

with $-1 < m < l \le 1$, $\tau = \frac{1-\sqrt{5}}{2}$, and $z \in \mathbb{E}$. For more details, see [9].

2. PROPOSED WORK

Inspired by the concept of cardioid region studied by Kumar and Kamaljeet [19] and Al-Shaikh *et al.* [22], we consider the class $\mathcal{A}(w)$ consists of analytic functions of the form:

$$f(z) = (z - w) + \sum_{n=2}^{\infty} a_n (z - w)^n \qquad (9)$$

normalized by the conditions $f(w) = 0$, $f'(w) = 1$, where w is a fixed point, investigated by Kanas and Ronning [23]. Based on these ideas, we now introduce the following new subclass of analytic functions of complex order γ.

Definition 1. A function $f(z) \in \mathcal{A}(w)$ of the form (9) is said to be in the class $\mathfrak{R}_{Car}(\gamma, \propto, w; z)$ of analytic functions of complex order γ provided that the following conditions hold:

$$1 + \frac{1}{\gamma}\left\{\frac{(z-w)f'(z) + \propto (z-w)^2 f''(z)}{(1-\propto)f(z) + \propto (z-w)f'(z)} - 1\right\} \prec \varphi_c(z),$$

where, $0 \leq \propto \leq 1$ and $\gamma \in \mathbb{C} \setminus \{0\}$ and $\varphi_c(z)$ is given by:

$$\varphi_c(z) = 1 + (z - w)e^{(z-w)};\ (z \in \mathbb{E}) \tag{10}$$

By selecting particular values for the parameters $\gamma, \propto$ and w in Definition 1, we obtain a known subclass and some new subclasses as special cases.
(i) $\mathfrak{R}_{Car}(\gamma, 0, w; z) \equiv \mathcal{S}^*_{Car}(\gamma, w; z)$: A new subclass of starlike functions characterized by a complex parameter γ, involving the cardioid domain.
(ii) $\mathfrak{R}_{Car}(1,0,0;z) \equiv \mathfrak{S}^*_{\mathcal{P}}$: Subclass of starlike functions connected with cardiod domain, introduced by Kumar and Kamaljeet [19].
(iii) $\mathfrak{R}_{Car}(\gamma, 1, w; z) \equiv \mathcal{C}_{Car}(\gamma, w; z)$: A new subclass of convex functions of complex γ associated with cardioid domain
(iv) $\mathfrak{R}_{Car}(1,1,w;z) \equiv \mathcal{C}_{Car}(w;z)$: A new subclass of w −convex functions connected to the cardioid domain.

3. SET OF LEMMAS

We make use of the lemmas stated below to derive our main results:
Lemma 1. If $p \in \mathcal{P}$ and it has the form (2), then

$$|c_n| \leq 2\ for\ n \geq 1, \tag{11}$$

Furthermore, for any $v \in \mathbb{C}$,

$$|c_2 - vc_1^2| \leq 2\,max\{1,\ |2v - 1|\} \tag{12}$$

For the inequality in (11), see [4], while the inequality in (12) was reported by Keogh and Merkes [24].

For the purpose of obtaining our primary results, it is necessary to prove the following lemma:
Lemma 2. Let
$p(z) = 1 + c_1(z - w) + c_2(z - w)^2 + \cdots,$
and $p(z) \prec \varphi_c(z)$, where φ_c is given by (10), then

$$|p_1| \leq 1,\quad |p_2| \leq 1\ and\ |p_3| \leq \frac{1}{2}$$

Results are sharp.

Proof. If $p(z) \prec \varphi_c(z)$, then there exists an analytic function $u(z)$ such that $|u(z)| \leq |z|$ in $\mathbb{E}$ and $p(z) = \varphi_c(u(z))$. Therefore, the function:

$$h(z) = \frac{1 + u(z)}{1 - u(z)}$$

$$= 1 + c_1(z - w) + c_2(z - w)^2 + c_3(z - w)^3 + \cdots,$$

is in the class $\mathcal{P}(0)$. It follows that

$$u(z) = \frac{c_1(z - w)}{2} + \left(c_2 - \frac{c_1^2}{2}\right)\frac{(z - w)^2}{2} + \left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right)\frac{(z - w)^3}{2} + \cdots,$$

therefore,

$$\varphi_c(u(z)) = 1 + u(z)e^{u(z)}.$$

Based on the series expansion of $u(z)$ and elementary computations, we have

$$\varphi_c(u(z)) = 1 + \frac{c_1}{2}(z - w) + \frac{c_2}{2}(z - w)^2 + \left(-\frac{c_1^3}{16} + \frac{c_3}{2}\right)(z - w)^3 + \left(\frac{c_1^4}{24} - \frac{3c_1^2c_2}{16} + \frac{c_4}{2}\right)(z - w)^4 + \cdots$$

$$= 1 + p_1(z - w) + p_2(z - w)^2 + p_3(z - w)^3 + p_4(z - w)^4 \ldots$$

Equating coefficients, we get:

$$p_1 = \frac{c_1}{2} \tag{13}$$

$$p_2 = \frac{c_2}{2} \tag{14}$$

and

$$p_3 = \left(-\frac{c_1^3}{16} + \frac{c_3}{2}\right) \tag{15}$$

The desired results can be obtained by applying Lemma $1(11)$ to equations $(13) - (15)$.

4. MAIN RESULTS

In this section, we determine the best possible bounds for the initial three coefficients and provide sharp estimates of the Fekete-Szegö inequality for the functions $f \in \mathfrak{R}_{Car}(\gamma, \propto, w; z)$. Moreover, to emphasize the connection between prior research and the new contributions, the well-known class is also highlighted.

Theorem 1. Let $f \in \mathfrak{R}_{Car}(\gamma, \propto, w; z)$, and be given by (9). Then,

$$|a_2| \le \frac{\gamma}{(1+\propto)}, \qquad |a_3| \le \frac{\gamma(1+\gamma)}{2(1+2\propto)}$$

$$and \quad |a_4| \le \frac{\gamma(\gamma^2+3\gamma+1)}{6(1+\propto)}.$$

These findings are sharp.

Proof. For given $f \in \Re_{Car}(\gamma, \propto, w; z)$, define

$$p(z) = 1 + p_1(z-w) + p_2(z-w)^2 + p_3(z-w)^3 + \cdots,$$

by

$$1 + \frac{1}{\gamma}\left\{\frac{(z-w)f'(z) + \propto (z-w)^2 f''(z)}{(1-\propto)f(z) + \propto (z-w)f'(z)} - 1\right\} = p(z),$$

where $p(z) \prec \varphi_c(z)$ in $\mathbb{E}$, and is given by (10). Hence,

$$1 + \frac{1}{\gamma}\left\{\frac{(z-w)f'(z) + \propto (z-w)^2 f''(z)}{(1-\propto)f(z) + \propto (z-w)f'(z)} - 1\right\}$$

$$= 1 + \frac{(1+\propto)a_2}{\gamma}(z-w)$$

$$+ \frac{(2(1+2\propto)a_3 - (1+\propto)^2 a_2^2)}{\gamma}(z-w)^2$$

$$+ \frac{(3(1+\propto)a_4 - 3(1+\propto)(1+2\propto)a_2 a_3 + (1+\propto)^3 a_2^3)}{\gamma}(z-w)^3 + \cdots$$

$$= 1 + p_1(z-w) + p_2(z-w)^2 + p_3(z-w)^3 + \cdots.$$

Equating coefficients of $(z-w)$, $(z-w)^2$, and $(z-w)^3$ we get:

$$a_2 = \frac{\gamma}{(1+\propto)} p_1 \tag{16}$$

$$a_3 = \frac{\gamma}{2(1+2\propto)} p_2 + \frac{\gamma^2}{2(1+2\propto)} p_1^2 \tag{17}$$

and

$$a_4 = \frac{\gamma}{3(1+\propto)} p_3 + \frac{\gamma^2}{2(1+\propto)} p_1 p_2 + \frac{\gamma^3}{6(1+\propto)} p_1^3 \tag{18}$$

By employing Lemma 2 on Equations (16) – (18), the required results are obtained. The function:

$$\mathfrak{H}_1(z) = z\, exp(e^z - 1) \tag{19}$$

plays the role of an extremal function when $\gamma = 1$ and $w = \propto = 0$, for which the equality holds.

Remark 1. For $w = 0, \gamma = 1$ and $\propto = 0$ in Theorem 1, we get the following known result, investigated by Kumar and Kamaljeet [19].

Theorem 2. If $f \in \mathfrak{S}_{\mathfrak{p}}^*$ and has the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then

$$|a_2| \le 1, \qquad |a_3| \le 1 \ \ and \ |a_4| \le \frac{5}{6}$$

These bounds are sharp.

Remark 2. For $\propto = 0$ in Theorem 1, we obtain the following new result for the subclass $\mathcal{S}_{Car}^*(\gamma, w; z)$.

Theorem 3. Let $f \in \mathcal{S}_{Car}^*(\gamma, w; z)$,where f is given by (9), then

$$|a_3| \le \frac{\gamma(1+\gamma)}{2},$$

$$|a_4| \le \frac{\gamma(\gamma^2+3\gamma+1)}{6}.$$

Remark 3. When $\propto = 1$, Theorem 1 reduces to the following new result for the subclass $\mathcal{C}_{Car}(\gamma, w; z)$.

Theorem 4. If $f \in \mathcal{C}_{Car}(\gamma, w; z)$ and is of the form (9), then

$$|a_2| \le \frac{\gamma}{2},$$

$$|a_3| \le \frac{\gamma(1+\gamma)}{6},$$

and

$$|a_4| \le \frac{\gamma(\gamma^2+3\gamma+1)}{12}.$$

Remark 4. When $\gamma = 1$ and $\propto = 1$, Theorem 1 leads to the following new result for the subclass $\mathcal{C}_{Car}(w; z)$, as follows.

Theorem 5. Let $f \in \mathcal{C}_{Car}(w; z)$, where f is given by (9), then

$$|a_2| \le \frac{1}{2}, \ |a_3| \le \frac{1}{3}, \ and \ |a_4| \le \frac{5}{12}.$$

Next, we turn our attention to the Fekete-Szegö problem for the subclass $\Re_{Car}(\gamma, \propto, w; z)$.

Theorem 6. If a function f of the form (9) belongs to $\Re_{Car}(\gamma,\propto,\mathcal{w};z)$, then

$$|a_3-\lambda a_2^2|\le$$

$$\frac{|\gamma|}{2(1+\propto)}\,max\left\{1,\left|\frac{2\gamma\lambda(1+2\propto)-(\gamma+1)(1+\propto)^2}{(1+\propto)^2}\right|\right\}$$

This result is sharp.

Proof. If $f\in\Re_{Car}(\gamma,\propto,\mathcal{w};z)$, then from (13) and (16), we get

$$a_2=\frac{\gamma c_1}{2(1+\propto)}\,.$$

Also, from $(13)-(14)$ and (17), we have

$$a_3=\frac{\gamma c_2}{4(1+2\propto)}+\frac{\gamma^2c_1^2}{8(1+2\propto)}.$$

This implies that

$$a_3-\lambda a_2^2=$$

$$\frac{\gamma}{4(1+2\propto)}\left\{c_2-c_1^2\left(\frac{2\gamma\lambda(1+2\propto)-\gamma(1+\propto)^2}{2(1+\propto)^2}\right)\right\}$$

Let,

$$v=\left(\frac{2\gamma\lambda(1+2\propto)-\gamma(1+\propto)^2}{2(1+\propto)^2}\right)$$

This leads to:

$$|a_3-\lambda a_2^2|=\frac{|\gamma|}{4(1+2\propto)}|c_2-vc_1^2|$$

Applying lemma $1(12)$ for v, we obtained the required result. For $\gamma=1$ and $\mathcal{w}=\propto=0$, equality cases hold for the function given by (1) when $\lambda\in\left[\frac{1}{2},\frac{2}{3}\right]$, and for $\mathfrak{H}_2(z)=z\,exp\left(\frac{e^z-1}{2}\right)$ when $\lambda\le\frac{1}{2}$ or $\lambda\ge\frac{2}{3}$. These are special cases of the general function $\mathfrak{H}_n(z)=z\,exp\left(\frac{e^{z^n}-1}{n}\right)$.

Remark 5. For $\gamma=1$ and $\propto=\mathcal{w}=0$ in Theorem 6, we obtain the known result for the subclass $\mathfrak{S}_{\mathcal{P}}^*$, investigated by Kumar and Kamaljeet [19].

Theorem 7. If a function f of the form (1) belongs to $\mathfrak{S}_{\mathcal{P}}^*$, then

$$|a_3-\lambda a_2^2|=\frac{1}{4}\left|c_2-\left(\lambda-\frac{1}{2}\right)c_1^2\right|\le\frac{1}{2}max(1,2|\lambda-1|).$$

The function for which equality holds is given by (19). In particular, for $\lambda=1$, we have:

$$|a_3-a_2^2|\le\frac{1}{2}.$$

Remark 6. For $\propto=0$ in Theorem 6, we obtain the following new result for the subclass $\mathcal{S}_{Car}^*(\gamma,\mathcal{w};z)$.

Theorem 8. If a function f of the form (9) belongs to $\mathcal{S}_{Car}^*(\gamma,\mathcal{w};z)$, then

$$|a_3-\lambda a_2^2|\le\frac{|\gamma|}{2}\,max\{1,|\gamma(2\lambda-1)-1|\}$$

This result is sharp.

Remark 7. The following new result for the subclass $\mathcal{C}_{Car}(\gamma,\mathcal{w};z)$ is obtained by taking $\propto=1$ in Theorem 6.

Theorem 9. Let $f\in\mathcal{C}_{Car}(\gamma,\mathcal{w};z)$ and be of the form (9). Then

$$|a_3-\lambda a_2^2|\le\frac{|\gamma|}{6}\,max\left\{1,\left|\frac{\gamma(3\lambda-4)-4}{4}\right|\right\}$$

Remark 8. The new result for the subclass $\mathcal{C}_{Car}(\mathcal{w};z)$ is obtained by setting $\gamma=1$ and $\propto=1$ in Theorem 6, as follows:

Theorem 10. Let $f\in\mathcal{C}_{Car}(\mathcal{w};z)$ be given (9). Then,

$$|a_3-\lambda a_2^2|\le\frac{1}{6}\,max\left\{1,\left|\frac{3\lambda-4}{2}\right|\right\}$$

5. CONCLUSIONS

In the present work, we introduce a new Ma–Minda type subclass $\Re_{Car}(\gamma,\propto,\mathcal{w};z)$ of univalent functions. Furthermore, by assigning specific values to the parameters γ, $\propto$ and $\mathcal{w}$, we define three additional subclasses: $\mathcal{S}_{Car}^*(\gamma,\mathcal{w};z)$ and $\mathcal{C}_{Car}(\gamma,\mathcal{w};z)$, $\mathcal{C}_{Car}(\mathcal{w})$, corresponding to starlike and convex functions, respectively. These subclasses are introduced in connection with the concepts of subordination and the cardioid domain. The fundamental geometric properties of these subclasses were investigated, including sharp bounds for the first three coefficients and sharp estimates for the Fekete–Szegö problem. Moreover, a well-known class and its associated results have been identified to demonstrate the connection between earlier work and the present investigation.

6. CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

7. REFERENCES

1. C. Tom (Ed.). Univalent Functions: The Basics (Chapter 8). In: Geometric Function Theory. *Springer, Cham, Switzerland* pp. 203-225 (2024).
2. A.W. Goodman (Ed.). Univalent functions. Vol. I. *Mariner Publishing Company* (1983).
3. A.W. Goodman (Ed.). Univalent functions. Vol. II. *Mariner Publishing Company* (1983).
4. C. Pommerenke (Ed.). Univalent Functions. *Vandenhoeck and Ruprecht, Göttingen* (1975).
5. D.K. Thomas, N. Tuneski, and A. Vasudevarao (Eds.). Univalent functions. Part of: De Gruyter Studies in Mathematics. *De Gruyter, Berlin* (2018).
6. P.L. Duren (Ed.). Univalent functions. Part of the book series: Grundlehren der mathematischen Wissenschaften (Volume 259). *Springer-Verlag, New York* (1983).
7. N.E. Cho, V. Kumar, and V. Ravichandran. A survey on coefficient estimates for Caratheodory functions. *Applied Mathematics E-Notes* 19: 370-396 (2019).
8. V. Kumar, N.E. Cho, V. Ravichandran, and H.M. Srivastava. Sharp coefficient bounds for starlike functions associated with the Bell numbers. *Mathematica Slovaca* 69(5): 1053-1064 (2019).
9. Y.J. Sun, M. Arif, L. Shi, and M.I. Faisal. Some further coefficient bounds on a new subclass of analytic functions. *Mathematics* 11(12): 2784 (2023).
10. M. Fekete and G. Szegö. Eine Bemerkung uber ungerade schlichte Funktionen. *Journal of the London Mathematical Society* 8: 85-89 (1933).
11. S. Kanas and H.E. Darwish. Fekete-Szegö problem for starlike and convex functions of complex order. *Applied Mathematics Letters* 23: 777-782 (2010).
12. A. Ahmed, A.W. Shaikh, and S.A. Shah. Fekete-Szegö Problem for Certain Analytic Functions of Complex Order Associated with Cardioid Domain. *International Journal of Analysis and Applications* 23: 109-109 (2025).
13. W. Ma and D. Minda. A unified treatment of some special classes of functions. In: Proceedings of the Conference on Complex Analysis. *Nankai Institute of Mathematics, Nankai University, Tianjin, China* pp. 157-169 (1992).
14. W. Janowski, Some extremal problems for certain families of analytic functions I. *Annales Polonici Mathematici* 28(3): 297-326 (1973).
15. M.S. Robertson. On the theory of univalent functions. *Annals of Mathematics* 37(2): 374-408 (1936).
16. N.E. Cho, V. Kumar, S.S. Kumar, and V. Ravichandran. Radius problems for starlike functions associated with the sine function. *Bulletin of the Iranian Mathematical Society* 45: 213-232 (2019).
17. M.K. Aouf, J. Dziok, and J. Sokol. On a subclass of strongly starlike functions. *Applied Mathematics Letters* 24(1): 27-32 (2011).
18. J. Sokoł and J. Stankiewicz. Radius of convexity of some subclasses of strongly starlike functions. *Zeszyty Naukowe Politechniki Rzeszowskiej. Matematyka* 19: 101-105 (1996).
19. S.S. Kumar and G. Kamaljeet. A cardioid domain and starlike functions. *Analysis and Mathematical Physics* 11(2): 34 (2021).
20. K. Sharma, N.K. Jain, and V. Ravichandran. Starlike functions associated with a cardioid. *Afrika Matematika* 27: 923-939 (2016).
21. V. Ravichandran, P. Yasar, B. Metin, and S. Arzu. Certain subclasses of starlike and convex functions of complex order. *Hacettepe Journal of Mathematics and Statistics* 34: 9-15 (2005).
22. S.B. Al-Shaikh, K. Matarneh, A.A. Abubaker, and M.F. Khan. Sharp Coefficient Bounds for a New Subclass of Starlike Functions of Complex Order γ Associated with Cardioid Domain. *Mathematics* 11(9): 2017 (2023).
23. S. Kanas and F. Ronning. Uniformly starlike and convex functions and other related classes of univalent functions. *Annales Universitatis Mariae Curie-Skłodowska. Sectio A* 53: 95-105 (1999).
24. F.R. Keogh and E.P. Merkes. A coefficient inequality for certain classes of analytic functions. *Proceedings of the American Mathematical Society* 20: 8-12 (1969).